Notes on representations of finite groups

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Let G be a finite group, F a field. We do not *assume F is algebraically closed unless explicitly stated, but we do assume always that* $\operatorname{char} F = 0$.

Contents

1 Representations

1.1 The group algebra

Definition 1. An ***F-representation*** *of G* is a finite-dimensional F-vector space V equipped with a group homomorphism $G \to \mathrm{GL}\,V$. In other words, V is a G-set whose action is given by linear transformations.

A ***morphism*** *of representations* is an F-linear map which commutes with the associated group action, i.e. $\varphi\colon V \to W$ satisfying $\varphi(gv) = g\varphi(v)$ for all $v \in V, g \in G$; we say in this case that the F-linear map φ is ***G-linear***, and write $\mathrm{Hom}_G(V, W)$ for the set of G-linear maps, as distinguished from the set of all F-linear maps $\mathrm{Hom}_F(V, W)$.

Hence we get a category $\mathsf{Rep}_F\, G$. The group G and field F remaining fixed, we will consistently drop F and G from the notation, referring e.g. to "representations" and the category Rep.

Definition 2. We define FG, the ***group algebra*** *of G over F*, to be the F-vector space whose basis consists of a formal symbols $[g]$ for each $g \in G$, together with an F-algebra structure defined by the multiplication of G. That is, we define $[g][h] := [gh]$ and extend this product F-linearly to the whole of FG. From now on we will drop the brackets and view G as sitting inside FG.

The reader should immediately convince themselves that Rep is equivalent (actually, isomorphic!) to the category of finite-dimensional FG-modules. We will alternate at will between viewing a representation as a vector space equipped with a group homomorphism and as an FG-module.

Remark 3. Morally, the perspective that "a representation is a vector space V equipped with a homomorphism $G \to \mathrm{GL}\,V$" is more correct than "a representation is an FG-module", even though these two categories are isomorphic. The reason is that the point of the theory we will set up is to study the group G. However, G is not determined by the algebra FG (Prop. 29), so we lose something crucial, namely the knowledge of the underlying group, when passing from $\mathsf{Rep}_F\, G$ to Mod_{FG}. A better perspective is to view FG as an F-algebra *with a distinguished basis* corresponding to the embedding $G \hookrightarrow FG$ (cf. Exc. 4). All this will be emphasized even more when we study characters and character tables, which are objects associated to representations and depend on the action of G, not just that of FG.

Exercise 4. There's a obvious bijection $FG \to \mathrm{Hom}_{\mathsf{Set}}(G, F)$, the latter being the set of functions $G \to F$. What is the corresponding algebra operation on $\mathrm{Hom}_{\mathsf{Set}}(G, F)$? What is the left action of G on $\mathrm{Hom}_{\mathsf{Set}}(G, F)$?

1.2 Examples of representations

Definition 5. Let $V \in \mathsf{Rep}$. A ***subrepresentation*** of V is an FG-submodule, i.e. an F-subspace $W \subseteq V$ such that $gW \subseteq W$ for all g.

Example 6. The algebra FG, when viewed as a left-module over itself, is called the ***regular representation***.

The ***trivial representation*** is just a copy of F with the trivial action of G ($ga = a$ for all $a \in F$). We'll use F to denote the trivial representation.

Example 7. The previous two examples can be seen as a special case of the following. Suppose G acts on a finite set X. Then we can form a vector space FX whose basis is the symbols $[x]$ for each $x \in X$ and let G act on FX in the obvious way: $g[x] := [gx]$. Such a representation FX is called a ***permutation representation***. Usually, as with FG, we'll omit brackets. Notice that $\sum_x x$ generates a trivial representation $F \subseteq FX$, and we can write $FX = F \oplus W$, where $W := \{\sum_x a_x x \colon \sum_x a_x = 0\}$. Both V and W are stable under the action of G and hence are subrepresentations.

Exercise 8. How do the orbits of X further decompose the representation FX?

Exercise 9. Show that if $g \in G$, then $\mathrm{tr}\, g|_{FX}$ is the number of elements of X fixed by g. Here, $g|_{FX}$ means "g acting on FX", i.e. the image of g under the associated group homomorphism $G \to \mathrm{GL}\, FX$.

Definition 10. Define $V^G := \{v \in V : gv = v \text{ for all } g\}$, a subrepresentation of V called the ***invariant subrepresentation*** *of* V, or the ***G-invariants*** *of* V. Obviously, V^G is a direct sum of trivial representations.

Example 11. Let $V \in \mathsf{Rep}$ and $v \in V$. Summing over G produces an invariant element: $\sum gv \in V^G$. Even better, the *average* over G, $(\#G)^{-1} \sum g \in FG$, defines a G-linear projection $V \twoheadrightarrow V^G$. (We will consistently write $\sum$ to abbreviate $\sum_{g \in G}$.)

Here are some examples of how to build new representations from old ones:

Example 12. Let $V, W \in \mathsf{Rep}$. Then $\mathrm{Hom}_F(V, W)$ can be made into a representation by ${}^g f(v) := gf(g^{-1}v)$. Observe that $\mathrm{Hom}_F(V, W)^G = \mathrm{Hom}_G(V, W)$ under this definition.

In particular, V^* (the dual vector space) is a representation via ${}^g\xi(v) = \xi(g^{-1}v)$.

Example 13. Let $V, W \in \mathsf{Rep}$. Then $V \otimes_F W$ becomes a representation by putting $g(v \otimes w) := gv \otimes gw$. Observe that this makes the natural isomorphism of vector spaces $V^* \otimes_F W \xrightarrow{\sim} \mathrm{Hom}_F(V, W)$ into a G-linear isomorphism; this will be important later.

Exercise 14. Generalize Ex. 13: show that if $U, V, W \in \mathsf{Rep}$, the map $\mathrm{Hom}_F(U, V) \otimes_F W \to \mathrm{Hom}_F(U, V \otimes_F W)$ given by $(f \otimes w)(u) := f(w) \otimes u$ is G-linear.

Example 15. Let H be another group, $V \in \mathsf{Rep}\, G$, and $W \in \mathsf{Rep}\, H$. Then $V \otimes_F W$ becomes a $G \times H$-representation by $(g, h)(v \otimes w) := gv \otimes hw$.

1.3 Irreducible representations, Maschke's theorem

Given a representation, how can we break it up into simpler subrepresentations?

Definition 16. Let $V \in \mathsf{Rep}$. We say V is ***irreducible*** if its only subrepresentations are 0 and V, and moreover $V \neq 0$.

Proposition 17. *Let $V \in \mathsf{Rep}$, and let $W \subseteq V$ be a subrepresentation. Then $V = W \oplus W'$ for some subrepresentation $W' \subseteq V$.*

Proof. Let $f : V \twoheadrightarrow W$ denote any projection map of F-vector spaces. Define $\varphi := (\#G)^{-1} \sum {}^g f$, an element of $\mathrm{Hom}_F(V, W)^G = \mathrm{Hom}_G(V, W)$. Moreover, $\varphi|_W = \mathrm{id}$:

$$\varphi(w) = \frac{1}{\#G} \sum gf(g^{-1}w) \overset{(*)}{=} \frac{1}{\#G} \sum gg^{-1}w = w.$$

Here, $(*)$ holds since f is a projection and $g^{-1}w \in W$. □

Corollary 18 (Maschke's theorem)**.** *Every finite-dimensional representation is a direct sum of irreducible representations.* □

Hence the situation is as nice as possible: to classify the representations of G, it suffices to classify the irreducible representations.

Remark 19. Every irreducible representation is isomorphic a quotient of FG by a maximal left ideal. Indeed, if V is such a representation, then for any $v \in V \setminus \{0\}$, its orbit FGv is a nonzero subrepresentation of V, hence equals V, so the surjection $FG \twoheadrightarrow FGv$ presents V as a quotient of FG. Cor. 18 implies that each irreducible representation also occurs as a *minimal left ideal* of FG, since given a maximal left ideal $V \subseteq FG$, we can write $FG \cong V \oplus W$ (as representations), so $FG/V \cong W$, i.e. the irreducible representation FG/V is isomorphic to the ideal W.

A useful special case of Cor. 18 is the following:

Corollary 20. *Let $f \colon V \to V$ be a linear map of F-vector spaces such that $f^n = \mathrm{id}$. Assume F is algebraically closed. Then T is diagonalizable.*

Proof. By Cor. 18, this is the same as showing that every irreducible representation of $\langle f \rangle$ (a finite cyclic group) is one-dimensional, which holds since f has an eigenvector (by our assumption $F = \overline{F}$). □

Exercise 21. Generalize the previous by showing that if G is Abelian and $F = \overline{F}$, then every irreducible representation is one-dimensional. Thus the irreducible representations of G constitute the ***Pontryagin dual group*** $\widehat{G} \coloneqq \mathrm{Hom}_{\mathsf{Grp}}(G, S^1)$, which by the structure theory of finite Abelian groups we know to be isomorphic to G.

Example 22. We study the irreducible representations of S_3 assuming $F = \overline{F}$. Let $\rho, \tau \in S_3$ be generators satisfying the relations $\rho^3 = \tau^2 = 1$ and $\tau = \rho\tau\rho$, and let V be an irreducible representation of FG. Assume $\dim V \geq 2$, so we can find linearly independent eigenvectors $v, w \in V$ for ρ, say $\rho v = \zeta_3^m v, \rho w = \zeta_3^n v$, where ζ_3 is a fixed 3rd root of unity. The second relation tells us that τv is an eigenvector for ρ with eigenvalue ζ_3^{-n}, since $\tau v = \rho\tau\rho v = \zeta_3^n \rho\tau v$. Thus either $m = -n$ and $\tau v = aw, \tau w = bv$ for some $a, b \in F$, or $m = n = 0$; but the latter forces V to be a representation of $S_3/\rho \cong C_2$, hence 1-dimensional. We conclude that aside from the two 1-dimensional irreducible representations of S_3, there is one more, which is 2-dimensional, given by the matrices

$$\rho \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \quad \tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(replacing v by bv allows us to assume $a = b = 1$). One sees that this V is irreducible by observing that there are no S_3-stable 1-dimensional subspaces.

Exercise 23. Let V denote the 2-dimensional irreducible representation of S_3 over $\mathbb{C}$ obtained in Ex. 22. The matrices obtained for ρ and τ show that V can be defined over $\mathbb{Q}(\zeta_3)$, i.e. there exists $V' \in \mathsf{Rep}_{\mathbb{Q}(\zeta_3)} S_3$ and an isomorphism $V' \otimes_{\mathbb{Q}(\zeta_3)} \mathbb{C} \cong V$ of $\mathbb{C}S_3$-representations.

(*a*) Let $V_{\mathbb{R}}$ be the representation arising from the action of S_3 on $\mathbb{R}^2$ as symmetries of an equilateral triangle. Show that $V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong V$.

(*b*) Let $V_{\mathbb{Q}}$ be the representation over $\mathbb{Q}$ arising from the sum-zero vectors in the permutation representation corresponding to the action of S_3 on $\{1, 2, 3\}$. Show that $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \cong V$.

We conclude from (b) that V can be defined over $\mathbb{Q}$.

Exercise 24. Generalize the argument of Ex. 22 to any dihedral group.

1.4 Schur's lemma and consequences

The following is one of those results which is important enough to canonically carry a venerated mathematician's name while being a complete triviality.

Lemma 25 (Schur's lemma)**.** *Let $V, W \in \mathsf{Rep}$.*

(a) *If V is irreducible, then any G-linear $\varphi\colon V \to W$ is injective or zero.*

(b) *If W is irreducible, then any G-linear $\varphi\colon V \to W$ is surjective or zero.*

(c) *If V and W are irreducible and $V \not\cong W$, then* $\operatorname{Hom}_G(V, W) = 0$.

(d) *If $F = \overline{F}$ and V is irreducible, then* $\operatorname{Hom}_G(V, V) = F \operatorname{id}$.

Proof. Parts (a), (b) are clear since kernels and images are subrepresentations, and (c) is immediate from them. For (d), given $\varphi\colon V \to V$ a G-linear map, it (as an F-linear map) has, since $F = \overline{F}$, an eigenvalue a; now apply (a) to $\varphi - a \operatorname{id}$. □

This gives a quick explanation for Exc. 21: if G is Abelian and V is an irreducible representation of G, each $g\colon V \to V$ is G-linear, hence multiplication by a scalar.

For convenience, we introduce the following possibly silly notation. Write P (or P_G, or P_{FG}) for the set of isomorphism classes of irreducible representations, and given $\lambda \in P$, let V_λ denote any representative of the class λ. Maschke's theorem reduces the problem of classifying the representations of G to describing the set P.

Schur's lemma suggests that we introduce the following pairing on Rep.

Definition 26. Given $V, W \in \mathsf{Rep}$, define $[V, W] \coloneqq \dim \operatorname{Hom}_G(V, W)$. For $\lambda \in P$, let $s_\lambda \coloneqq [V_\lambda, V_\lambda]$.

This pairing allows us to immediately reap quite a harvest from Maschke's theorem and Schur's lemma. The statement of Prop. 27 is again longer than its proof, but it should not be underestimated.

Proposition 27.

(a) *If $\lambda, \mu \in P$ are distinct, then $[V_\lambda, V_\mu] = 0$. If $F = \overline{F}$, then $s_\lambda = 1$ for all λ.*

(b) *The pairing $[-, -]$ is symmetric and biadditive with respect to $\oplus$.*

(c) *Let $V \in \mathsf{Rep}$. Then $V \cong \bigoplus_\lambda V_\lambda^{n_\lambda(V)}$, where $n_\lambda(V) = [V, V_\lambda]/s_\lambda$. In particular, the decomposition of V into irreducibles is unique up to isomorphism and reordering.*

(d) *Let $V \in \mathsf{Rep}$. If $[V, V] = 1$, then V is irreducible. If $F = \overline{F}$, then V is irreducible if and only if $[V, V] = 1$.*

(e) *$FG \cong \bigoplus_\lambda V_\lambda^{(\dim V_\lambda)/s_\lambda}$ as representations.*

(f) $\sum_\lambda (\dim V_\lambda)^2/s_\lambda = \#G$.

Proof. Part (a) rephrases Lem. 25(c, d). Biadditivity of $[-,-]$ holds since $\oplus$ is a categorical biproduct, and symmetry then follows since $[V_\lambda, V_\mu] = [V_\mu, V_\lambda]$ for all $\lambda, \mu \in P$. Part (c) follows from Cor. 18, biadditivity, and the values of $[V_\lambda, V_\mu]$ for $\lambda, \mu \in P$, and (d) is a special case. Then (c) implies (e) since $\mathrm{Hom}_G(FG, V) \cong V$ for all V. Finally, (f) follows from comparing dimensions in (e). □

In particular, (f) implies that P is always finite, and in fact that $\#P \leq \#G$, but we later will be more precise (Cors. 30 and 42).

Exercise 28. Let $V \in \mathsf{Rep}$, $V \cong \bigoplus_\lambda V_\lambda^{n_\lambda}$. Show that $[V, V] = \sum_\lambda s_\lambda n_\lambda^2$. Then prove that the following are equivalent:

(a) V is irreducible.

(b) $[V, V]$ divides $[V, W]$ for all $W \in \mathsf{Rep}$.

(c) Either $[V, W] = 0$ or $[V, V] \leq [V, W]$ for all $W \in \mathsf{Rep}$.

Proposition 29. *The natural map $FG \to \bigoplus_\lambda \mathrm{End}_F V_\lambda$ of F-algebras is injective. If $F = \overline{F}$, it is an isomorphism.*

Proof. By Prop. 27(a), (e), the dimension of both sides agree if $F = \overline{F}$, so it suffices to prove injectivity. Assume $\alpha \in FG$ maps to zero. Then α acts by zero on every irreducible representation, hence by zero on *every* representation by Cor. 18, in particular FG itself. But then $\alpha \cdot 1 = 0$, so $\alpha = 0$, as desired. □

Let G_{class} denote the set of conjugacy classes of G. Then $Z(FG) = FG_{\text{class}}$, the latter denoting those sums whose coefficients are constant on conjugacy classes (here Z means center). The elements of FG_{class} correspond naturally to class functions on G, i.e. functions $F \to G$ constant on conjugacy classes. Combining this with Prop. 29 and recalling that the center of a matrix algebra is composed of the scalar matrices gives:

Corollary 30. *Assume $F = \overline{F}$. Then $\#P = \#G_{\text{class}}$.* □

Exercise 31. Prove the converse to Exc. 21: if $F = \overline{F}$, then G is Abelian if and only if every irreducible representation is one-dimensional.

Exercise 32. Interpret Prop. 29 as not a map of F-algebras but as a $G \times G$-linear map in such a way that taking $G \times G$-invariants is the same as taking center.

Exercise 33. Recall that a group is *perfect* if its Abelianization is trivial. Show that there is no perfect group of order 24.

2 Characters

2.1 Definition, first properties

Assume for the moment that $F = \overline{F}$. The isomorphism $FG \to \bigoplus_\lambda \mathrm{End}_F V_\lambda$ is defined in an obvious way, but what is less obvious is what its inverse should look like. One should hope that the induced isomorphism $FG_{\text{class}} \cong \bigoplus_\lambda F \operatorname{id}_{V_\lambda}$ can be rephrased as a natural correspondence between irreducible representations and class functions on G. We certainly know, by the classical identity $\operatorname{tr}(XY) = \operatorname{tr}(YX)$, of *a* correspondence.

Namely, no longer assuming $F = \overline{F}$, we define:

Definition 34. Let $V \in \mathsf{Rep}$. The ***character*** *of* V is the class function χ_V defined by $\chi_V(g) := \operatorname{tr} g|_V$. We say that χ_V is an ***irreducible character*** if V is irreducible; we'll abbreviate $\chi_\lambda := \chi_{V_\lambda}$ for $\lambda \in P$.

The values of a character lie in the field F, but since the eigenvalues of a finite-order operator are roots of unity, they actually lie in the subfield of F consisting of those elements algebraic over $\mathbb{Q}$, which itself can be identified with a subfield of $\overline{\mathbb{Q}} \subseteq \mathbb{C}$. (Even better, the values, being integral combinations of roots of unity, lie in the ring of integers of a cyclotomic extension of $\mathbb{Q}$.) Thus, fixing such an identification, we may think of characters as being $\mathbb{C}$-valued.

Example 35. When $F = \overline{F}$ and G is Abelian, the "character table" of G is easy to write down. One simply has to decompose G as a direct sum of cyclics and allow the values of a generator g to vary over the roots of unity whose exponent divides the order of g. For example, if $G = \langle g \rangle \oplus \langle h \rangle$ where g, h have order 2, the characters correspond to all the functions $\{g, h\} \to \{\pm 1\}$:

G	e	g	h	gh
$(1, 1)$	1	1	1	1
$(1, -1)$	1	1	-1	-1
$(-1, 1)$	1	-1	1	-1
$(-1, -1)$	1	-1	-1	1

Here the columns vary over the conjugacy classes of G, the rows vary over the irreducible representations of G, and each entry shows the value of an irreducible character on a conjugacy class. Clearly such tables, for G Abelian and $F = \overline{F}$, are highly structured, as already known to Dirichlet. We will see that much of this structure carries over to the non-Abelian case.

Exercise 36. Write down the character tables corresponding to Ex. 22 and Exc. 24.

Lemma 37. *Let* $V, W \in \mathsf{Rep}$ *and* $g \in G$.

(*a*) $\chi_{V \oplus W} = \chi_V + \chi_W$.

(*b*) $\chi_{V \otimes_F W} = \chi_V \chi_W$.

(*c*) $\chi_{V^*}(g) = \overline{\chi}_V(g)$.

Proof. These all follow from moving up to an algebraic closure of F and writing out eigenbases using Cor. 20. □

Exercise 38. Let $V \in \mathsf{Rep}$. Then $V \otimes_F V$ has subrepresentations $\bigwedge^2 V$ and $\operatorname{Sym}^2 V$ spanned, respectively, by elements of the form $v \otimes w - w \otimes v$ and $v \otimes w + w \otimes v$. Express $\chi_{\bigwedge^2 V}$ and $\chi_{\operatorname{Sym}^2 V}$ in terms of χ_V. What happens when you generalize 2 to n?

2.2 Pairing of class functions, orthogonality

We would like to transfer our pairing $[-,-]$ from representations to class functions using the association $\chi\colon \mathsf{Rep} \to FG_{\text{class}}$, i.e. find a pairing $[-,-]$ on FG_{class} with the property that $[\chi_V, \chi_W] = [V, W]$ for all $V, W \in \mathsf{Rep}$. (Why do this? Aside from our goal to study the relationship between class functions and representations, we saw above how useful $[-,-]$ is—showing that it can be computed in terms of class functions imposes more structure and makes it even easier to work with.)

Only a little modification of the naïve inner product on FG_{class} is needed:

Proposition 39. *Let $V, W \in \mathsf{Rep}$. Then $(\#G)^{-1} \sum \chi_V(g)\overline{\chi}_W(g) = [V, W]$.*

Proof. Combining Lem. 37 and Ex. 13 gives

$$\sum \chi_V(g)\overline{\chi}_W(g) = \sum \chi_V(g)\chi_{W^*}(g) = \sum \chi_{V\otimes_F W^*}(g) = \sum \chi_{\mathrm{Hom}_F(V,W)}(g);$$

the latter is $\operatorname{tr} \sum g|_{\mathrm{Hom}_F(V,W)}$ by definition of χ. But $(\#G)^{-1} \sum g$ is projection onto the invariant subspace $\mathrm{Hom}_F(V,W)^G = \mathrm{Hom}_G(V,W)$, which has dimension $[V,W]$. $\square$

Definition 40. Let c, d be class functions. We define $[c, d] := (\#G)^{-1} \sum c(g)\overline{d}(g)$.

Translating Prop. 27(c), (d) into the language of characters via Prop. 39 gives a very computationally convenient way to compute the multiplicity of an irreducible representation in a given representation or test whether a given representation is irreducible.

Furthermore, translating Lem. 25(c), (d) and Cor. 30 into the language of characters gives:

Corollary 41. *The irreducible characters form an orthogonal system in FG_{class} with respect to $[-,-]$; if $F = \overline{F}$, they form an orthonormal basis.* $\square$

Corollary 42. *The inequality $\#P \leq \#G_{\text{class}}$ holds even when $F \neq \overline{F}$.* $\square$

Exercise 43. Assuming $F = \overline{F}$, use Cor. 41(b) to show that $V \otimes_F W$ is irreducible whenever $V, W \in \mathsf{Rep}$, V irreducible, $\dim W = 1$. Can you find a proof for the case $F \neq \overline{F}$ using Exc. 28 (restated in the language of characters)? Can you find a proof which doesn't use characters?

There is another useful orthogonality relation when $F = \overline{F}$. Namely, not only are the rows of a character table orthogonal (Cor. 41(a)), but so are its columns:

Proposition 44. *Assume $F = \overline{F}$. Then*

$$\sum_\lambda \chi_\lambda(g)\overline{\chi}_\mu(g) = \begin{cases} \#G/\#C(g), & \mu = \lambda \\ 0, & \mu \neq \lambda, \end{cases}$$

for all $g \in G$, where $C(g)$ denotes the conjugacy class of g in G.

Proof. We rewrite the orthogonality of the rows (Cor. 41) as a matrix equation. Namely, let $\chi_1, \dots, \chi_n$ be the irreducible characters and let $g_1, \dots, g_n$ be representatives for the conjugacy classes of G. Let $X := (\chi_i(g_j))_{i,j}$ and $Y := (c_i\overline{\chi}_j(g_i))_{i,j}$, where $c_i := \#C(g)/\#G$. Then $XY = I$, and the claim follows from the identity $YX = I$. $\square$

Finally, before presenting some concrete consequences of our theory of characters, we answer our question about the inverse of the map $FG \to \bigoplus_\lambda \operatorname{End}_F V_\lambda$, which motivated the study of characters in the first place. To do this, we use the orthogonality of characters to generalize the projection map $(\#G)^{-1}\sum g$.

Proposition 45. *Assume $F = \overline{F}$, and let $\lambda, \mu \in P$. Then $(\#G)^{-1}\sum \chi_\lambda(g)g^{-1}$ acting on V_μ is multiplication by $(\dim V_\lambda)^{-1}$.*

Proof. Take the trace of the given operator using Prop. 39; it lies in $FG_{\text{class}} = Z(FG)$, hence defines a G-linear map, so we may apply Lem. 25(d). $\square$

Thus an "orthogonal idempotent" of $\bigoplus_\lambda \operatorname{End}_F V_\lambda$, i.e. an element of the form $(0, \dots, 0, \mathrm{id}_{V_\lambda}, 0, \dots, 0)$, corresponds to $\varepsilon_\lambda := (\dim V_\lambda)(\#G)^{-1}\sum \chi_\lambda(g)g^{-1} \in FG$. When F is not algebraically closed, the elements $\varepsilon_\lambda := (\dim V_\lambda)(s_\lambda \#G)^{-1}\sum \chi_\lambda(g)g^{-1}$ still satisfy $\varepsilon_\lambda\varepsilon_\mu = 0$ if $\lambda \neq \mu$, but are not necessarily idempotent.

2.3 Examples

Example 46. We compute the character table of S_4. We know of two easy irreducible representations: the trivial representation F, the 1-dimensional "sign" representation F_{sign} corresponding to the sign homomorphism $S_4 \to F^\times$. The permutation representation V of S_4 acting on $\{1, 2, 3, 4\}$ breaks up as $V = F \oplus V_{\text{std}}$ as seen in Ex. 7, and the character of V_{std} (the "standard representation") is known: compute χ_V using Exc. 9 and use that $\chi_{\text{std}} + \chi_{\text{triv}} = \chi_V$ (Lem. 37(a)). One sees from this that V_{std} is irreducible and $V_{\text{std}} \otimes_F F_{\text{sign}}$ is distinct from V_{std} (use Lem. 37(b)).

Cor. 41(c) and Prop. 27(f) show that there is exactly one missing irreducible representation W. Assume now $F = \overline{F}$; then Prop. 27(f) further tells us it is 2-dimensional. Putting everything together, our character table now looks like

	1	6	8	6	3
S_4	e	$(1\,2)$	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(1\,2)(3\,4)$
F	1	1	1	1	1
F_{sign}	1	-1	1	-1	1
V_{std}	3	1	0	-1	-1
$V_{\text{std}} \otimes_F F_{\text{sign}}$	3	-1	0	1	-1
W	2	?	?	?	?

Here, we have put at the top of each column the size of the corresponding conjugacy class. There are several ways to proceed from here. We describe three approaches below, from least to most ad hoc (dually, from least to most conceptual).

(a) The sum of the squares of the dimensions of the irreducible representations is the order of S_4, so we easily see that $\chi_W(e) = \dim W = 2$. We could now use orthogonality of the rows to set up a linear system of four equations in four variables, or orthogonality of the columns (Prop. 44) to solve for each missing entry individually. As a shortcut, we must have $\chi_W(1\,2) = \chi_W(1\,2\,3\,4) = 0$, since $\chi_W \chi_{\text{sign}} = \chi_W$ (there are exactly 5 irreducible representations, which forces $W \otimes F_{\text{sign}} \cong W$).

(b) We compute the character of $W_0 \coloneqq \operatorname{Sym}^2 V_{\text{std}}$ using the $\frac{1}{2}(\chi(g)^2 + \chi(g^2))$ formula from Exc. 38:

$$W_0 \;\big|\; 6 \quad 2 \quad 0 \quad 0 \quad 2$$

One now checks $\langle \chi_{W_0}, \chi_F \rangle = \langle \chi_{W_0}, \chi_{V_{\text{std}}} \rangle = 1$, hence $W_0 = W \oplus V_{\text{std}} \oplus F$. Computing the character of this W gives the missing row.

(c) Observing that we have an isomorphism

$$S_4/\{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} \xrightarrow{\sim} S_3,$$

we may start with the standard representation of S_3 and pull it back, via the quotient map, to a representation W of S_4. Then W is a two-dimensional irreducible representation of S_4, for if $W = W_1 \oplus W_2$, then W_1 and W_2 would be stable under the action of S_3 (since the action of any element of S_3 is the same as the action of some element of S_4, namely any preimage under the quotient map). So the character of W is the missing row, and this is easy to fill in since we know the character of the standard representation for S_3.

The last two approaches have the advantage of not requiring the assumption $F = \overline{F}$ since they show that W is defined over F. We complete table thus:

	1	6	8	6	3
S_4	e	$(1\,2)$	$(1\,2\,3)$	$(1\,2\,3\,4)$	$(1\,2)(3\,4)$
F	1	1	1	1	1
F_{sign}	1	−1	1	−1	1
V_{std}	3	1	0	−1	−1
$V_{\text{std}} \otimes_F F_{\text{sign}}$	3	−1	0	1	−1
W	2	0	−1	0	−2

Exercise 47. Compute the character tables of A_4 and (for the masochist) S_5.

Exercise 48. ($F = \overline{F}$). Show that the character table of the quaternion group is the same as that of D_4. (*Recommendation:* Isolate the common structure of the two groups which completely determines the table, i.e. compute them both at once.)

Finally, we give a general calculation which we feel emphasizes the power of the theory set up so far. Let us begin by considering the standard representation V_{std} of S_n, defined as in Ex. 22(b) and the first paragraph of Ex. 46. We saw in these examples that

V_{std} is irreducible for $n = 3, 4$, but it is not trivial to prove this in general (especially without characters—try it!). In the following proposition, we will go much further and, without too much difficulty, classify the irreducible (up to a single trivial summand) permutation representations of any group.

Proposition 49. *Let G be a group acting on a set X and V the associated permutation representation, and form the usual decomposition $V = F \oplus W$. Then W is irreducible if and only if G acts doubly-transitively on X.*

Recall that "doubly-transitively" means that for any $x, x', y, y' \in X$ with $x \neq y$ and $x' \neq y'$, one can find $g \in G$ with $gx = x'$ and $gy = y'$.

Proof. By Cor. 41(b), it suffices to show that $[\chi_V, \chi_V] = 2$. By definition,

$$[\chi_V, \chi_V] = \frac{1}{\#G} \sum_{g \in G} \chi_V(g)^2,$$

and by Exc. 9, $\chi_V(g)$ is the number of fixed points of the action of g on X. Observe now that $\chi_V(g)^2$ is then the number of fixed points of the action of g on $X \times X$. That is, $[\chi_V, \chi_V] = (\#G!)^{-1} \sum_{g \in G} \#(X \times X)^g$. Hence by Burnside's counting lemma, $[\chi_V, \chi_V]$ is the number of orbits of $X \times X$ under G. There are two obvious G-stable sets, the diagonal $\{(a, b) \in X \colon a = b\}$ and its complement, and both these sets are orbits if and only if the action is doubly transitive. $\square$