## Triangulated categories

Let $\mathcal{A}$ be an abelian category. You may not use roofs in the first three exercises, as (some of) these results are used to prove that roofs work in $D(\mathcal{A})$.

**Exercise 18.** Suppose $f\colon A \to B$ is a map of complexes in $\mathrm{Kom}(\mathcal{A})$. Construct maps $\alpha_f\colon B \to \mathrm{Cyl}(f)$ and $\beta_f\colon \mathrm{Cyl}(f) \to B$ in $\mathrm{Kom}(\mathcal{A})$ that give a homotopy equivalence $B \cong \mathrm{Cyl}(f)$. Moreover, show that $\alpha_f$ and $\beta_f$ are inverses in $D(\mathcal{A})$ (without invoking the fact that the map to the derived category factors through the homotopy category).

**Exercise 19.** Suppose $f\colon A \to B$ is a map of complexes in $\mathrm{Kom}(\mathcal{A})$. Show that we have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\bar{f}"] & \mathrm{Cyl}(f) \arrow[r] & C(f) \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & B \arrow[r] \arrow[u, "\alpha_f"] & C(f) \arrow[r] & A[1] \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

**Exercise 20.** Suppose $f, g\colon A \to B$ are homotopic.

(a) Using a homotopy $h$ between $f$ and $g$, construct a map $H\colon \mathrm{Cyl}(f) \to \mathrm{Cyl}(g)$ so that

$$\begin{tikzcd}
 & \mathrm{Cyl}(f) \arrow[dd, "H"] \\
A \arrow[ur, "\bar{f}"] \arrow[dr, "\bar{g}"'] & \\
 & \mathrm{Cyl}(g)
\end{tikzcd}$$

commutes, where $\bar{f}, \bar{g}$ are as in the previous exercise.

(b) Show that $\beta_g \circ H \circ \alpha_f = \mathrm{id}_B$.

(c) Show that $\beta_g \circ \bar{g} = g$.

(d) Show that $\bar{f} = \alpha_f \circ f$ in $D(\mathcal{A})$.

(e) Conclude that $f = g$ in $D(\mathcal{A})$.

**Exercise 21.** Recall that we defined $\mathrm{Ext}^i(A, B) := \mathrm{Hom}_{D(\mathcal{A})}(\widetilde{A}, \widetilde{B}[i]))$. Give a bijection between elements of $\mathrm{Ext}^1(A, B)$ and extensions of $A$ by $B$, i.e. equivalence classes of exact sequences $0 \to B \to C \to A \to 0$.

**Exercise 22.** Suppose $\mathcal{D}$ is a triangulated category. Show that for any object $U \in \mathcal{D}$, the functor $\mathrm{Hom}(U, -)$ is *cohomological* in the sense that for an exact triangle $A \to B \to C \to A[1]$, the sequence

$$\cdots \to \mathrm{Hom}(U, C[-1]) \to \mathrm{Hom}(U, A) \to \mathrm{Hom}(U, B) \to \mathrm{Hom}(U, C) \to \mathrm{Hom}(U, A[1]) \to \cdots$$

is exact.