Pretalk 1: Chow groups

Exercise 1. Suppose X is a finite type separated k-scheme, with closed subscheme $\iota : Z \hookrightarrow X$. Show that there is an exact sequence

$$\mathrm{CH}_i(Z) \xrightarrow{\iota_*} \mathrm{CH}_i(X) \xrightarrow{j^*} \mathrm{CH}_i(X \setminus Z) \to 0,$$

where $j : X \setminus Z \hookrightarrow X$.

Exercise 2. Let $X \subseteq \mathbb{P}^n$ be a subvariety. Show that $[X] \neq 0$ in $\mathrm{CH}^*(\mathbb{P}^n)$. Hint: First show it for a point $p \in \mathbb{P}^n$.

Exercise 3. Show that

$$\mathrm{CH}^*(\mathbb{P}^n) = \frac{\mathbb{Z}[h]}{(h^{n+1})},$$

where h is the class of a hyperplane. Moreover, show that any codimension i linear subspace of $\mathbb{P}^n$ has class h^i.

Exercise 4. Consider $X = V(x_1, x_2) \cup V(x_3, x_4) \subseteq \mathbb{P}^4$. Show that $[X] \cdot h^2$ is not equal to $[X \cap V(x_1 - x_3, x_2 - x_4)]$ (despite the fact that the latter has the appropriate dimension).

Exercise 5. Say a finite type separated k-scheme X has the Chow–Künneth generation property (CKgP) if for all Y, the Künneth map

$$\mathrm{CH}(X) \otimes \mathrm{CH}(Y) \to \mathrm{CH}(X \times Y)$$

is an isomorphism. Note $\mathbb{A}^n$ has the CKgP by the homotopy invariance property of Chow groups. Show that $\mathbb{P}^n$ has the CKgP.

Exercise 6. Let E be a genus one curve. Show that the image of $\mathrm{CH}(E) \otimes \mathrm{CH}(E) \to \mathrm{CH}(E \times E)$ does not contain the class of the diagonal (hence E does not have the CKgP). Hint: Use the adjunction formula.

Exercise 7. Let X be an equidimensional finite type separated k-scheme of positive dimension. Show that given $\alpha, \beta \in Z^0(X)$, there exists $\alpha' \in Z^0(X)$ such that $\alpha' \sim \alpha$ and $|\alpha'| \cap |\beta| = \emptyset$.