Chow motives

Fix a field k. Before stating the exercises, we give some reminders on correspondences and Chow motives.

Definition. Let $X, Y \in \mathbf{SmProj}_k$. We set $\mathrm{Corr}(X,Y) \coloneqq \mathrm{CH}(X \times Y)$, and define a grading $\mathrm{Corr}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Corr}^i(X,Y)$ by $\mathrm{Corr}^i(X,Y) \coloneqq \mathrm{CH}^{\dim(Y)+i}(X\times Y)$. Elements of $\mathrm{Corr}(X,Y)$ are called *correspondences* from X to Y, and are said to be of *degree* i if they come from $\mathrm{Corr}^i(X,Y)$.

Recall also that correspondences may be composed; if $\gamma \in \mathrm{Corr}(X,Y)$ and $\delta \in \mathrm{Corr}(Y,Z)$, it is given by

$$\delta \circ \gamma \coloneqq \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\gamma) \cdot \mathrm{pr}_{23}^*(\delta)),$$

where pr_{ij} is the projection onto the (i,j) factor of $X \times Y \times Z$. This makes $\mathrm{Corr}^*(X,X)$ into an associative graded algebra; it has a unit element, the class of diagonal $[\Delta_X]$, which we will denote by 1 below (hopefully not causing confusion). Moreover, $\mathrm{Corr}^*(X,Y)$ becomes a graded $\mathrm{Corr}^*(Y,Y)$-$\mathrm{Corr}^*(X,X)$-bimodule.

Definition. The category $\mathbf{CHM}_k$ of Chow motives over k consists of triples (X,π,m) with $X \in \mathbf{SmProj}_k$, $\pi \in \mathrm{Corr}^0(X,X)$ an idempotent correspondence (meaning $\pi\circ\pi = \pi$), and $m \in \mathbb{Z}$. The hom sets are

$$\mathrm{Hom}_{\mathbf{CHM}_k}((X,\pi,m),(Y,\rho,n)) \coloneqq \rho \circ \mathrm{Corr}^{n-m}(X,Y)\circ \pi.\text{[7]}$$

Finally, for any ring R, let $\mathbf{CHM}_k \otimes R$ denote the category of Chow motives *with coefficients in* R: the definition is the same as that of $\mathbf{CHM}_k$, but we replace $\mathrm{Corr}^*(X,Y)$ with $\mathrm{Corr}^*(X,Y)\otimes R$ everywhere.

Notation:

- We have a functor $\mathfrak{h}\colon \mathbf{SmProj}_k \to \mathbf{CHM}_k$ given by $\mathfrak{h}(X) \coloneqq (X,1,0)$ and $\mathfrak{h}(f) \coloneqq [\Gamma_f^{\mathrm{t}}]$ for $X \in \mathbf{SmProj}_k$ and a morphism f in $\mathbf{SmProj}_k$.
- We set $\mathbf{1}(m) \coloneqq (\mathrm{Spec}(k),1,m)$.
- Let $M \coloneqq (X,\pi,m) \in \mathbf{CHM}_k$. Given an idempotent morphism $\pi'\colon M \to M$ in $\mathbf{CHM}_k$, we set $\pi' M \coloneqq (X,\pi',m)$. Given $n\in\mathbb{Z}$, we set $M(n) \coloneqq (X,\pi,m+n)$.
- If $M \in \mathbf{CHM}_k$, let $M\otimes R$ denote the corresponding object of $\mathbf{CHM}_k\otimes R$.

Exercise 23. The purpose of this exercise is to do some simple calculations with Chow motives. Let E be an elliptic curve over k with identity element $O \in E(k)$. Recall that we defined $\pi^0, \pi^1, \pi^2 \in \mathrm{Corr}^0(E,E)$ by

$$\pi^0 \coloneqq [\{O\}\times E], \quad \pi^2 \coloneqq [E\times\{O\}], \quad \pi^1 \coloneqq 1 - \pi^0 - \pi^2.$$

We saw that $1 = \pi^0+\pi^1+\pi^2$ is a decomposition of the identity morphism $\mathfrak{h}(E)\to\mathfrak{h}(E)$ into orthogonal idempotents, which yields a decomposition

$$\mathfrak{h}(E) = \mathfrak{h}^0(E)\oplus\mathfrak{h}^1(E)\oplus\mathfrak{h}^2(E),$$

[7]Note that the identity morphism $(X,\pi,m)\to(X,\pi,m)$ is represented by the cycle $\pi = \pi\circ 1\circ\pi \in \pi\circ\mathrm{Corr}^0(X,X)\circ\pi$.

where $\mathfrak{h}^i(E) \coloneqq \pi^i \mathfrak{h}(E)$.

We are going to try to further decompose $\mathfrak{h}(E)$, at least after extending scalars. Namely, let $f\colon E \to E$ be the inversion map, and consider the following elements of $\mathrm{Corr}^0(E,E) \otimes \mathbb{Z}[\frac{1}{2}]$:

$$\pi_f^+ \coloneqq \frac{1+\mathfrak{h}(f)}{2}, \quad \pi_f^- \coloneqq \frac{1-\mathfrak{h}(f)}{2}.$$

(a) Show that $\mathfrak{h}(E) \otimes \mathbb{Z}[\frac{1}{2}] = \pi_f^+ \mathfrak{h}(E) \oplus \pi_f^- \mathfrak{h}(E)$. (In the notation, we have implicitly extended scalars to $\mathbb{Z}[\frac{1}{2}]$ on the right-hand side.)

(b) Show that

$$\pi_f^+ \mathfrak{h}(E) = \left(\mathfrak{h}^0(E) \otimes \mathbb{Z}[\tfrac{1}{2}]\right) \oplus \pi_f^+ \mathfrak{h}^1(E) \oplus \left(\mathfrak{h}^1(E) \otimes \mathbb{Z}[\tfrac{1}{2}]\right),$$

while $\pi_f^- \mathfrak{h}(E) = \pi_f^- \mathfrak{h}^1(E)$.

(c) Let H^* be a Weil cohomology theory over k. Use the Lefschetz trace formula to show that f^* acts as multiplication by -1 on $H^1(E)$. Deduce that $H(\pi_f^+ \mathfrak{h}^1(E)) = 0$. (Thus $\pi_f^+ \mathfrak{h}^1(E)$ is a "phantom motive"—it has trivial cohomology in any theory.)

(d) Show that, in fact, π_f^+ is rationally equivalent to 0, hence $\pi_f^+ \mathfrak{h}^1(E) = 0$. (Of course, this implies the result of (c), but it's still a good exercise to work out Lefschetz's trace formula.)

Exercise 24. Assume k is algebraically closed and is not an algebraic extension of a finite field. In this exercise, you will prove that $\mathbf{CHM}_k \otimes \mathbb{Q}$ is not an Abelian category[8] in the following steps. (Below, we sometimes omit the $\otimes \mathbb{Q}$ from the notation.)

(a) Let E be an elliptic curve over k, and let $P \in E(k)$ be a non-torsion point. Using the cycle $[P] - [O] \in \mathrm{CH}^1(E)$, construct nonzero morphisms $\varphi\colon \mathfrak{h}(E) \to \mathbf{1}$ and $\psi\colon \mathbf{1}(-1) \to \mathfrak{h}(E)$ in $\mathbf{CHM}_k \otimes \mathbb{Q}$.

(b) Show that $\psi \circ (\varphi(-1))\colon \mathfrak{h}(E)(-1) \to \mathfrak{h}(E)$ is the zero map. (*Hint:* Show that it is represented by a cycle in $\mathrm{CH}^2(E \times E)$ of the form

$$\Big((P,P) + (O,O) - 2(Q,Q)\Big) + \Big(2(Q,Q) - (P,O) - (O,P)\Big),$$

and choose $Q \in E(k)$ to make both summands rationally equivalent to 0.)

(c) Conclude that $\mathbf{1}$ admits a proper subobject M.

(d) According to [Deligne–Milne, *Tannakian Categories*, Proposition 1.17], if $\mathbf{CHM}_k \otimes \mathbb{Q}$ were Abelian, then $\mathbf{1} = M \oplus M'$ for some M'. Why is this impossible?

Exercise 25. Let $\mathbf{AM}_k$ be the category of *Artin motives* over k, i.e. the full subcategory of $\mathbf{CHM}_k \otimes \mathbb{Q}$ whose elements have the form (X, π, m) with $\dim(X) = 0$. Construct an equivalence of categories $\mathbf{AM}_k \cong \mathbf{Rep}_{\mathbb{Q}}^{\mathrm{cts}}(\mathrm{Gal}_k)$.[9]

[8] In the lecture, we saw that if $\mathbf{CHM}_k \otimes \mathbb{Q}$ is a *semisimple* Abelian category, then rational equivalence equals numerical equivalence over k, which easily implies that k is an algebraic extension of a finite field.

[9] An object of $\mathbf{Rep}_{\mathbb{Q}}^{\mathrm{cts}}(\mathrm{Gal}_k)$ is by definition a finite-dimensional $\mathbb{Q}$-vector space equipped with a continuous $\mathbb{Q}$-linear action of Gal_k, where $\mathbb{Q}$ is given the discrete topology and Gal_k the Krull topology. Equivalently, such an action is continuous if it factors through $\mathrm{Gal}(k' \mid k)$ for some finite extension $k' \mid k$.