Pretalk 2: Sites

Exercise 8. Let X be a topological space, and let $\mathbf{Top}_X$ be the category of topological spaces over X.[1] The purpose of this exercise is to study the categories of sheaves on different Grothendieck topologies on $\mathbf{Top}_X$. Consider the sites

$$(\mathbf{Top}_X)_{\mathrm{all}}, \quad (\mathbf{Top}_X)_{\mathrm{surj}}, \quad (\mathbf{Top}_X)_{\text{ét}}, \quad (\mathbf{Top}_X)_{\mathrm{Zar}}$$

all with underlying category $\mathbf{Top}_X$ and whose coverings are as follows: a set $\{f_i \colon U_i \to U\}_{i\in I}$ of morphisms in $(\mathbf{Top}_X)$ with the same target is a covering in

- $(\mathbf{Top}_X)_{\mathrm{all}}$ no matter what.
- $(\mathbf{Top}_X)_{\mathrm{surj}}$ if it is *jointly surjective*, i.e. $U = \bigcup_{i\in I} f(U_i)$.
- $(\mathbf{Top}_X)_{\text{ét}}$ if it is jointly surjective and each f_i is a *local homeomorphism*, i.e. each point in U_i has an open neighborhood which f_i maps homeomorphically onto an open subset of U.
- $(\mathbf{Top}_X)_{\mathrm{Zar}}$ if it is jointly surjective and each f_i is an *open embedding*, i.e. f_i maps U_i homeomorphically onto an open subset of U.

After convincing yourself that each is in fact a site, show the following:

(1) The category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{all}}$ is equivalent to the category with one object and one morphism.
(2) The category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{surj}}$ is equivalent to the category of sets.
(3) The category of sheaves of sets on $(\mathbf{Top}_X)_{\text{ét}}$ and $(\mathbf{Top}_X)_{\mathrm{Zar}}$ are *equal*: a presheaf on $\mathbf{Top}_X$ is a sheaf on $(\mathbf{Top}_X)_{\text{ét}}$ if and only if it is on $(\mathbf{Top}_X)_{\mathrm{Zar}}$.

Exercise 9. Let X be a topological space.

(1) Show that for any $U \in \mathbf{Top}_X$, the representable functor $h_U \colon \mathbf{Top}_X^{\mathrm{op}} \to \mathbf{Set}$ given by $h_U(V) \coloneqq \mathrm{Hom}_{\mathbf{Top}_X}(V, U)$ is a sheaf on $(\mathbf{Top}_X)_{\mathrm{Zar}}$ but not $(\mathbf{Top}_X)_{\mathrm{surj}}$. (By the previous exercise, it is also a sheaf on $(\mathbf{Top}_X)_{\text{ét}}$, which is less obvious!)
(2) Conclude that the category of sheaves of sets on $(\mathbf{Top}_X)_{\mathrm{Zar}}$ need not be equivalent to the category of sheaves of sets on X (in the usual sense). (Hint: Take X to be a point.)
(3) Nevertheless, exhibit a fully faithful functor $B \colon \mathbf{Sh}(X, \mathbf{Set}) \hookrightarrow \mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{Zar}}, \mathbf{Set})$, and show that if $\mathcal{F}$ is a sheaf of Abelian groups on X, then its cohomology equals that of $B(\mathcal{F})$.

Exercise 10. Let $k \to A$ be a ring map, k a field. Prove that the following are equivalent:

(1) A is Noetherian, zero-dimensional, and every local ring $(A \otimes_k \overline{k})_{\mathfrak{p}}$ is regular.
(2) $A \cong \prod_{i\in I} k_i$ for some finite set I and finite separable extensions $k_i \mid k$.

(Hint: The structure theorem for Artinian rings might be useful; see Atiyah–Macdonald, Theorem 8.7.) Thus, under our definitions, "étale" is indeed equivalent to "smooth with zero-dimensional fibers".

Exercise 11. Let k be a field.

(1) Show that every presheaf on $\mathbf{\acute{E}t}_{\mathrm{Spec}(k)}$ is in fact a Nisnevich sheaf.
(2) Conclude that sheaf cohomology on the small Nisnevich site of $\mathrm{Spec}(k)$ vanishes in all positive degrees (unlike the small étale site).

[1]That is, an object of $\mathbf{Top}_X$ is a pair (Y, f) consisting of a topological space Y and a continuous map $f \colon Y \to X$; a morphism $(Y, f) \to (Z, g)$ in $\mathbf{Top}_X$ is a continuous map $h \colon Y \to Z$ satisfying $g \circ h = f$. Note that $\mathbf{Top}_X$ is (equivalent to) the category of all topological spaces if X is a point.

Exercise 12. Let **Sch** be the category of all schemes. On **Sch**, we have presheaves $\mathbb{G}_{\mathrm{m}}$ and μ_n given as follows:

$$\mathbb{G}_{\mathrm{m}}(U) \coloneqq \mathcal{O}_U(U)^{\times}, \quad \mu_n(U) \coloneqq \mathrm{Ker}(\mathcal{O}_U(U)^{\times} \xrightarrow{f \mapsto f^n} \mathcal{O}_U(U)^{\times}).$$

In the following, you may assume that $\mathbb{G}_{\mathrm{m}}$ is an étale sheaf (hence a Nisnevich or Zariski sheaf, hence also μ_n is an étale/Nisnevich/Zariski sheaf). The *Kummer sequence* is the following sequence of sheaves:

$$0 \to \mu_n \to \mathbb{G}_{\mathrm{m}} \xrightarrow{f \mapsto f^n} \mathbb{G}_{\mathrm{m}} \to 0.$$

(1) Show that the Kummer sequence is exact in the category of étale sheaves when restricted to the subcategory $\mathbf{Sch}_{\mathrm{Spec}(\mathbb{Z}[1/n])}$ (schemes U in which $n \in \mathcal{O}_U(U)^{\times}$).
(2) Show that the Kummer sequence is *not* exact in the category of Zariski sheaves, even when restricted to $\mathbf{Sch}_{\mathrm{Spec}(\mathbb{Z}[1/n])}$.
(3) What about the category of Nisnevich sheaves?
(4) What goes wrong with the Kummer sequence in the étale topology on the whole category **Sch**?

Exercise 13. Let X be a scheme, and let $x\colon \mathrm{Spec}(K) \to X$ be a point of X. An *étale neighborhood* of x is a commuting diagram of the form

$$\begin{array}{ccc} & \nearrow & U \\ & & \downarrow \\ \mathrm{Spec}(K) & \xrightarrow{x} & X \end{array}$$

in which $U \to X$ is étale. We abbreviate this diagram as "$(U, u) \to (X, x)$". The étale neighborhoods of x form a category in an obvious way.

Show that the category of étale neighborhoods of x is filtered. This means:

(1) Given $(U, u) \to (X, x)$ and $(U', u') \to (X, x)$, there exists a commutative diagram of the form

$$\begin{array}{ccc} (U'', u'') & \longrightarrow & (U', u') \\ \downarrow & & \downarrow \\ (U, u) & \longrightarrow & (X, x); \end{array}$$

(2) Given $f, g\colon (U, u) \to (X, x)$, there exists $h\colon (U', u') \to (U, u)$ with $h \circ f = g \circ f$.

Thus if $\mathcal{F}$ is a presheaf on $\mathbf{\acute{E}t}_X$, the filtered colimit

$$\mathcal{F}_x \coloneqq \operatorname*{colim}_{(U,u)\to(X,x)} \mathcal{F}(U)$$

can be computed in the usual way; it is called the *stalk* of $\mathcal{F}$ at x.