MOTIVES IN MAY

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0. Advertisement

Jake, April 11.

This section consists of notes for a talk given at the OSU TAGGS (*Topology, geometry, and Applications* Graduate Student Seminar).[1] The purpose was to advertise Motives in May to an audience with a background in topology, differential geometry, and analysis. Alas, only a small piece of what I would have liked to say fit in the allotted 50 minutes.

0.1. Cohomology from different perspectives

Let X be a topological space. One attaches to X various cohomology groups, for example, singular cohomology with coefficients in a ring B,

$$H^i_{\mathrm{sing}}(X; B),$$

whose elements we are going to view (by the universal coefficient theorem) as linear functions on the B-module of singular homology classes.

Now assume X is a smooth manifold. In this setting, we have another notion of cohomology, the *de Rham cohomology*

$$H^i_{\mathrm{dR}}(X),$$

which is defined to be the cohomology of the de Rham complex

$$0 \to \Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}} \cdots,$$

in which $\Omega^i(X)$ is the $\mathbb{R}$-vector space of C^∞ differential forms on X and d is the exterior derivative. This is of a very different flavor than the singular cohomology, but nonetheless Stokes's theorem tells us that integration of a form over a homology class defines an $\mathbb{R}$-linear map

$$\begin{aligned} H^i_{\mathrm{dR}}(X) &\longrightarrow H^i_{\mathrm{sing}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R} \\ [\omega] &\longmapsto ([\alpha] \mapsto \textstyle\int_\alpha \omega) \end{aligned} \tag{0.1.1}$$

(Here, ω is a C^∞ i-form on X, α is a cycle on X, and $[-]$ denotes cohomology/homology class.) The theorem of de Rham is that the map (0.1.1) is an *isomorphism*. Moreover, it is natural in X, i.e. is an isomorphism between the two functors

$$H^i_{\mathrm{dR}} \quad \text{and} \quad H^i_{\mathrm{sing}}(-; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$$

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[1] https://sites.google.com/view/osu-taggs/home

from the category of smooth manifolds to the category of $\mathbb{R}$-vector spaces. In this sense, despite being defined in a very different way, H^i_{dR} is not a "new mathematical object". Instead, it provides us with new perspectives, new ways to compute cohomology, and new connections between different branches of mathematics, all of which, of course, are extremely valuable.

Now assume that our smooth manifold X is compact, without boundary, and equipped with a Riemannian metric and an orientation. Hodge came up with the correct generalization of the Laplacian operator Δ in this setting; it acts on the groups $\Omega^i(X)$. Any differential form which is killed Δ is also closed (i.e. killed by d), so we get a map

$$\Omega^i(X)^{\Delta=0} \to H^i_{\mathrm{dR}}(X). \tag{0.1.2}$$

A major theorem of Hodge[2] is that (0.1.2) is an *isomorphism*. That is, every de Rham cohomology class on X has a unique representative ω satisfying $\Delta\omega = 0$; such a form ω is called *harmonic*. This opens up the possibility of studying the topology of X via analysis and differential equations.

0.2. The Hodge decomposition

Let X be a complex manifold. A $\mathbb{C}$-valued C^∞ differential form ω on X is called a (p,q)-*form* if for any point $x \in X$, there is a neighborhood U of x with holomorphic coordinate functions $(z_1, \ldots, z_p, w_1, \ldots, w_q)$ such that

$$\omega|_U = f\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p \wedge \mathrm{d}\overline{w_1} \wedge \cdots \wedge \mathrm{d}\overline{w_q}$$

for some C^∞ function $f\colon U \to \mathbb{C}$. It is not hard to see that any $\mathbb{C}$-valued C^∞ differential form may be uniquely decomposed into a sum of (p,q)-forms (for varying (p,q)). That is, we have a decomposition

$$\Omega^i(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=i} \Omega^{p,q}(X). \tag{0.2.1}$$

Assume now and for the rest of this talk that X is compact and admits a Kähler metric.[3] For example, if X can be embedded into complex projective space $\mathbb{P}^n(\mathbb{C})$, then X admits a Kähler metric. In this case, the (p,q)-components of a harmonic form are again harmonic, so the decomposition (0.2.1) descends, by Hodge's theorem, to a decomposition

$$H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=i} H^{p,q}(X).$$

This is the *Hodge decomposition*. It evidently satisfies *Hodge symmetry*:

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

We call $h^{p,q}(X) \coloneqq \dim_{\mathbb{C}}(H^{p,q}(X))$ the (p,q)-*Hodge number* of X.

The key thing to note here is that, in contrast to everything described in the previous sections, the complex de Rham cohomology of a compact Kähler manifold carries a *new*

[2] the proof of which had some gaps which were filled in by Kodaira and Weyl

[3] that is, a Hermitian metric—the complex-analytic analogue of a Riemannian metric—satisfying an extra technical condition

structure that was invisible before: the sequence of Betti numbers are refined to a grid of Hodge numbers, which are reflecting the complex geometry, rather than topology, of X.

To see that the Hodge numbers really give us something new, a simple consequence of Hodge symmetry is that the odd Betti numbers b^{2i+1} of a compact Kähler manifold must be even. So, for example, we immediately see that $S^1 \times S^3$ cannot be given the structure of a Kähler manifold.

0.3. Hodge structures

We can produce an even "better" invariant than the Hodge numbers by combining the Hodge decomposition with de Rham's theorem. This leads to the following definition.

0.3.1. Definition. Let $B \subseteq \mathbb{R}$ be a ring (typically $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$). A (*pure*) *B-Hodge structure* is a finitely generated B-module V together with a "Hodge" decomposition

$$V \otimes_B \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

satisfying $\overline{V^{p,q}} = V^{q,p}$ for all p, q.[4] These form a category B-**HS**, in which morphisms are homomorphisms of B-modules which respect the Hodge decompositions after tensoring with $\mathbb{C}$. We say that V is of *weight i* if $V^{p,q} = 0$ whenever $p + q \neq i$.

Explicitly, we get a functor from the category of compact Kähler manifolds to B-**HS** which sends X to the B-module $H^i_{\mathrm{sing}}(X; B)$ together with its Hodge structure induced by de Rham's theorem and the Hodge decomposition. Intuitively, the B-Hodge structure describes how the B-submodule $H^i_{\mathrm{sing}}(X; B) \subseteq H^i_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C}$ is "positioned relative to" the Hodge decomposition.

At a first glance, it is not at all obvious what we have gained (if anything) by making this definition. Nonetheless, the Hodge structure on $H^i_{\mathrm{sing}}(X; B)$ turns out to be extremely powerful, as the following example illustrates.

0.3.2. Example. Recall that every 1-dimensional complex manifold, i.e. Riemann surface, which is compact and of genus 1 is isomorphic to $\mathbb{C}/\Lambda$ for some lattice Λ. Moreover, $\mathbb{C}/\Lambda$ and $\mathbb{C}/\Lambda'$ are complex-diffeomorphic if and only if $\Lambda' = z\Lambda$ for some $z \in \mathbb{C}$. Of course, the Betti numbers of $\mathbb{C}/\Lambda$ are $(1, 2, 1)$, so its Hodge numbers are forced to be

$$\begin{array}{ccc} & h^{1,1} & \\ h^{0,1} & & h^{1,0} \\ & h^{0,0} & \end{array} \quad = \quad \begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

by Hodge symmetry. So the Hodge decomposition provides no information in this case.

On the other hand, one has an isomorphism of $\mathbb{Z}$-Hodge structures $H^1_{\mathrm{sing}}(\mathbb{C}/\Lambda, \mathbb{Z}) \cong \Lambda$, where Λ is given the following Hodge structure:

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{C} = (\Lambda \otimes_{\mathbb{Z}} \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C} =: \Lambda^{0,1} \oplus \Lambda^{1,0}.$$

[4] Since $B \subseteq \mathbb{R}$, we have $V \otimes_B \mathbb{C} = (V \otimes_B \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, each element of which maybe written uniquely as $v \otimes 1 + w \otimes i$ for some $v, w \in V \otimes_B \mathbb{R}$; this allows us to define complex conjugation on $V \otimes_B \mathbb{C}$.

It is not difficult to check that a $\mathbb{Z}$-module homomorphism $\Lambda \to \Lambda'$ of lattices in $\mathbb{C}$ respects Hodge decompositions (if and) only if becomes $\mathbb{C}$-linear after tensoring with $\mathbb{R}$, i.e. is multiplication by some $z \in \mathbb{C}$. Consequently, the Hodge structure on $H^1_{\text{sing}}(\mathbb{C}/\Lambda, \mathbb{Z})$ is a *complete* invariant of a compact Riemann surface of genus 1. In fact, the category of complex tori is equivalent to the category of free rank-2 $\mathbb{Z}$-Hodge structures with $h^{0,1} = h^{1,0} = 1$.

It is remarkable that a Hodge structure—just a bit of linear-algebraic data—is able to capture so much geometric information!

0.4. An enriched cohomology theory

For the rest of this talk, let us abbreviate $H^i(X)$ for $H^i_{\text{sing}}(X; B)$ endowed with its B-Hodge structure. Then H^i is an "enriched cohomology theory" in the sense that the B-Hodge structure compatible with the usual axioms of cohomology:

- If Y is another compact Kähler manifold, Künneth's theorem states that the natural map

$$H^*(X) \otimes_B H^*(Y) \to H^*(X \times Y) \tag{0.4.1}$$

  is an isomorphism. This is true *at the level of B-Hodge structures*: The category B-**HS** has tensor product, and it makes (0.4.1) into an isomorphism of B-Hodge structures.
- The Poincaré duality pairing

$$H^i(X) \otimes_B H^{2d-i}(X) \xrightarrow{\smile} H^{2d}(X) \tag{0.4.2}$$

  is again a morphism in B-**HS**; here d is the (complex) dimension of X.

Note that there is just one possibility for the Hodge structure $H^{2d}(X)$: We have $H^{2d}(X) \cong B(-d)$, where $B(-d)$ is the unique B-Hodge structure which is free of rank 1 and of weight $2d$. That is, its underlying B-module is just B, and $B(-d)^{d,d} = \mathbb{C}$ $(= B \otimes_B \mathbb{C})$. Thus (0.4.2) induces an isomorphism $H^i(X)(d) \cong H^{2d-i}(X)^\vee$ of B-Hodge structures, where for a B-Hodge structure V we write $V(d) \coloneqq V \otimes_B B(d)$ and $V^\vee$ for its dual as a B-Hodge structure. One calls $V(d)$ the dth "Tate twist" of V; its underlying B-module is just V, but the Hodge decomposition has been shifted in weight.

0.5. The Hodge conjecture

Each (possibly singular) closed complex submanifold Z of X gives rise to a class $[Z] \in H^{2i}(X)$, where i is the (complex) codimension of Z. Such a class will always live in $H^{i,i}(X)$, essentially because it arises from $H^0(Z)$, whose Hodge decomposition only has an $(0, 0)$-piece.

0.5.1. Conjecture. (The "Hodge conjecture".) Assume X is a smooth projective complex variety, and take $B = \mathbb{Q}$. Then the $\mathbb{Q}$-vector space $H^{2i}(X) \cap H^{i,i}(X)$ is spanned by classes of the form $[Z]$ for closed subvarieties Z of X of codimension i.

In the above conjecture, we are taking the intersection of the $\mathbb{Q}$-vector space $H^{2i}(X)$ with the $\mathbb{C}$-vector space $H^{i,i}(X)$ inside the $\mathbb{C}$-vector space $H_{\text{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C}$. It should be noted that the analogous statement is known to be false if we let X be an arbitrary compact Kähler manifold or if we replace $\mathbb{Q}$ by $\mathbb{Z}$.

The point of the rest of this section is to give a "motivic interpretation" of the Hodge conjecture.

The starting point is the observation that the category $\mathbb{Q}$-**HS** has an internal hom, because it has tensors and duals:

$$\underline{\mathrm{Hom}}(V, W) \coloneqq V^\vee \otimes_{\mathbb{Q}} W.$$

One can recover the usual hom as follows:

$$\mathrm{Hom}_{\mathbb{Q}\text{-}\mathbf{HS}}(V, W) = \underline{\mathrm{Hom}}(V, W) \cap \underline{\mathrm{Hom}}(V, W)^{0,0}.$$

Again, we are here intersecting the $\mathbb{Q}$-vector space $\underline{\mathrm{Hom}}(V, W)$ with the $\mathbb{C}$-vector space $\underline{\mathrm{Hom}}(V, W)^{0,0}$ inside $\underline{\mathrm{Hom}}(V, W) \otimes_{\mathbb{Q}} \mathbb{C}$.[5] Therefore, given to smooth projective complex varieties X and Y of dimensions d and e, respectively, we have

$$\begin{aligned} H^{e,e}(X \times Y) &= (H^*(X) \otimes_{\mathbb{Q}} H^*(Y))^{e,e} \\ &= (H^*(X) \otimes_{\mathbb{Q}} H^*(Y)(e))^{0,0} \\ &= (H^*(X) \otimes_{\mathbb{Q}} H^*(Y)^\vee)^{0,0} \\ &= \underline{\mathrm{Hom}}(H^*(Y), H^*(X))^{0,0}, \end{aligned}$$

the first equality being the "enriched" Künneth isomorphism, and the third being the "enriched" Poincaré duality. So the Hodge conjecture for $X \times Y$ says that $\mathrm{Hom}_{\mathbb{Q}\text{-}\mathbf{HS}}(H^*(Y), H^*(X))$ is spanned by classes of the form $[Z]$ for closed subvarieties Z of $X \times Y$ of codimension e. Of course, the graph of a morphism $f \colon X \to Y$ is such a subvariety, and indeed gives rise to a pullback map $f^* \colon H^*(Y) \to H^*(X)$ which is a morphism in $\mathbb{Q}$-**HS**.

Now let us be a bit vague. The Hodge conjecture suggests that we should consider *all* closed codimension-e subvarieties of $X \times Y$, and even $\mathbb{Q}$-linear combinations of such, to be "generalized morphisms" from X to Y. Suppose we have a category $\mathbf{Corr}_{\mathbb{C}}$ whose objects are smooth projective complex manifolds and whose morphisms are "generalized morphisms" in the sense of the previous sentence. Then the Hodge conjecture says that the functor

$$H^* \colon \mathbf{Corr}_{\mathbb{C}}^{\mathrm{op}} \to \mathbb{Q}\text{-}\mathbf{HS}$$

is *full*. Moreover, by imposing an appropriate equivalence relation $\sim$ on $\mathbf{Corr}_{\mathbb{C}}$, we obtain, under the Hodge conjecture, a *fully faithful* functor

$$H^* \colon (\mathbf{Corr}_{\mathbb{C}}/{\sim})^{\mathrm{op}} \to \mathbb{Q}\text{-}\mathbf{HS},$$

where we have quotiented the hom sets of $\mathbf{Corr}_{\mathbb{C}}$ by $\sim$. In other words: The Hodge conjecture says that the data of the $\mathbb{Q}$-Hodge structure $H^*(X)$ is precisely the data of X itself, viewed as an object in the category $\mathbf{Corr}_{\mathbb{C}}/{\sim}$! Moreover, the category $\mathbf{Corr}_{\mathbb{C}}$ is defined in an entirely

[5] It should be noted that $\mathrm{Hom}_{\mathbb{Q}\text{-}\mathbf{HS}}(V, W) = \mathrm{Hom}_{\mathbb{Q}\text{-}\mathbf{HS}}(\mathbb{Q}, \underline{\mathrm{Hom}}(V, W))$, where $\mathbb{Q}$ means $\mathbb{Q}(0)$ (the unit object for the tensor product in $\mathbb{Q}$-**HS**).

geometric way, and although $\sim$ is cohomological in nature, the Hodge conjecture implies that it too can be described in geometric terms (i.e. without cohomology).

0.6. Motives: A look forward

The "motivic interpretation" of the Hodge conjecture in the previous paragraph raises some questions (aside from how to make everything precise).

- It suggests that we should view the functor
$$\mathbf{SmProj}_{\mathbb{C}}^{\mathrm{op}} \to (\mathbf{Corr}_{\mathbb{C}}/\sim)^{\mathrm{op}}$$
as a sort of enriched cohomology theory for smooth projective complex varieties. However, the category $(\mathbf{Corr}_{\mathbb{C}}/\sim)^{\mathrm{op}}$ is a rather deficient receptacle for a cohomology theory: Although its hom sets are $\mathbb{Q}$-vector spaces, it is very far from being Abelian, and also does not have duals. Can we "improve" it to have these good properties (and more)? For example, can we somehow "adjoin kernels and cokernels"?
- How do we carry out these ideas for smooth projective varieties over an arbitrary field?

These are basic questions in the theory of *pure motives*, and have been to a large extent answered, though important (and foundational) open questions remain. Although the discussion above does not make evident that it would have any useful applications, a complete theory of pure motives would have far-reaching consequences in geometry, arithmetic, and even the theory of transcendental numbers.

The philosophy of *mixed motives* is that pure motives should be a special case of a theory that deals with *all* varieties, not just smooth projective ones. This theory remains much more shrouded in mystery, and attempts at developing it take on a very different flavor than the theory of pure motives.

1. Chow groups

Jake, April 15.

1.1. Cycles

From a first pass, Chow groups can be thought of as providing invariants of a scheme which describe its "shape" (in some sort of way that takes into account its algebraic structure). Recall that, in algebraic topology, the standard approach is to study certain subspaces of a topological space up to certain equivalences, e.g. loops up to homotopy, or simplices (more precisely, "cycles"—linear combinations of simplices) up to homology. We'll try to do something similar, which will end up being sort of like "cycles up to homotopy".

It turns out that the algebraic nature of schemes provides us with a very natural candidate for which "subspaces" to consider. For a scheme X, let $|X|$ denote its underlying (Zariski) topological space.

1.1.1. Exercise. If X is any scheme whatsoever, show that the association $x \mapsto \overline{\{x\}}$ defines a bijection

$$|X| \xrightarrow{\sim} \{\text{irreducible closed subsets of } |X|\}.$$

Also, show that every closed subset of $|X|$ is the underlying set of a unique reduced closed subscheme of X.

We are going to pass back and forth between these equivalent objects, hopefully without causing confusion. Perhaps most natural perspective from the point of view of topology is "irreducible closed subsets", but we will see that it is very useful to take advantage of their scheme structures.

For the rest of this talk, all schemes are of finite type over a fixed field k.[6] We will write $\mathbb{A}^n$ and $\mathbb{P}^n$ for $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$ and $X \times Y$ for $X \times_k Y$.

1.1.2. Definition. A *cycle* on X is a $\mathbb{Z}$-linear combination of integral[7] closed subschemes of X. For such a subscheme Z, we will write $[Z]$ for the associated cycle. Let $\mathcal{Z}(X)$ denote the (free) Abelian group of cycles on X. It has two obvious gradings:

(a) It has a grading *by dimension* (the "homological grading"): $\mathcal{Z}(X) = \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}_i(X)$, where $\mathcal{Z}_i(X)$ is the group of cycles of dimension i, meaning $\mathbb{Z}$-linear combinations of integral closed subschemes of X of dimension i. Elements of $\mathcal{Z}_i(X)$ are called i*-cycles*.

(b) If X is irreducible (hence, by smoothness, irreducible), $\mathcal{Z}(X)$ also has a well-defined grading *by codimension* (the "cohomological grading"): $\mathcal{Z}(X) = \bigoplus_{i=0}^{\dim(X)} \mathcal{Z}^i(X)$, where $\mathcal{Z}^i(X) \coloneqq \mathcal{Z}_{\dim(X)-i}(X)$. More generally, if $X = X_1 \sqcup \cdots \sqcup X_n$ with each X_i irreducible, then $\mathcal{Z}(X) = \mathcal{Z}(X_1) \oplus \cdots \oplus \mathcal{Z}(X_n)$ for obvious reasons, and we set $\mathcal{Z}^i(X) \coloneqq \mathcal{Z}^i(X_1) \oplus \cdots \oplus \mathcal{Z}^i(X_n)$. Note that, in this case, $\mathcal{Z}^i(X)$ need not equal $\mathcal{Z}_j(X)$ for any j, unless all X_i have the same dimension. Below, whenever we discuss the

[6] comment on generality

[7] Recall that "integral" is equivalent to "reduced and irreducible".

grading $\mathcal{Z}^*(X)$, we will implicitly assume that X is a disjoint union of irreducible schemes.

1.1.3. Example. $\mathcal{Z}^0(X)$ is freely generated by the set of connected components of X.

Cycles are defined in terms of *integral* closed subschemes of X, but it will be very useful to define a cycle attached to *any* closed subscheme of X; this will allow us to capture "multiplicity" information.

1.1.4. Recall. For a ring R and an R-module M, we say that M is of *finite length* if has a filtration whose graded pieces are simple R-modules. In this case, $\operatorname{len}_R(M)$ is defined to be the number of such graded pieces; it is well defined by the standard "Jordan–Hölder" argument. Otherwise, we set $\operatorname{len}_R(M) := \infty$.

1.1.5. Definition. Let Z be a closed subscheme of X. For each irreducible component W of Z, let $\mathcal{O}_{Z,W}$ denote the local ring at the generic point of W. Then we define

$$[Z] := \sum_W \operatorname{len}_{\mathcal{O}_{Z,W}}(\mathcal{O}_{Z,W}) \cdot [W], \tag{1.1.5.1}$$

where W ranges over all the (finitely many) irreducible components of Z.

1.1.6. Exercise. Complete the following to check that the definition makes sense.

(a) Let R be a Noetherian local ring. Show that if R has Krull dimension 0, then $\mathfrak{m}_R^n = 0$. (In fact, the converse is true too.)

(b) Let X be a locally Noetherian scheme, let Z be a closed subscheme of X, and let W be an irreducible component of Z. Show that $\operatorname{len}_{\mathcal{O}_{Z,W}}(\mathcal{O}_{Z,W}) < \infty$.

1.1.7. Remark. All coefficients appearing in (1.1.5.1) are positive, because $\mathcal{O}_{Z,W}$ is nonzero.

1.2. Rational equivalence and Chow groups

Just as in the development of singular homology, we have to take a quotient to get groups which (intrinsically) carry any nontrivial geometric meaning about X. We do this by trying to port the notion of "homotopy" to the algebraic setting. Intuitively, if we take an algebraic function (i.e. morphism of schemes) $f\colon X \to \mathbb{P}^1$, we view f as providing a "homotopy"—a nicely varying family—from $f^{-1}\{0\}$ to $f^{-1}\{\infty\}$. The actual definition modifies this intuition in three ways.

- We will use Definition 1.1.5 to capture the correct multiplicities (i.e. the vanishing/pole orders of f).
- We will take into account that $f^{-1}\{0\}$ and $f^{-1}\{\infty\}$ are always of codimension 1.
- Rather than looking only at morphisms to $\mathbb{P}^1$, we will allow ourselves more flexibility by working with any rational function (element of a function field).

1.2.1. Definition. Consider a pair (Z, f) consisting of an integral closed subscheme Z of X and an element $f \in k(Z)^\times$, where $k(Z) := \mathcal{O}_{Z,Z}$ is the function field of Z. Given an irreducible closed subset W of Z such that $\operatorname{codim}_Z(W) = 1$, we can write $f = g/h$ with $g, h \in \mathcal{O}_{Z,W}$, and we define the *order of vanishing* of (Z, f) along W to be

$$\operatorname{ord}_W(Z, f) := \operatorname{len}_{\mathcal{O}_{Z,W}}(\mathcal{O}_{Z,W}/g) - \operatorname{len}_{\mathcal{O}_{Z,W}}(\mathcal{O}_{Z,W}/h)$$

(where, as above, $\mathcal{O}_{Z,W}$ denotes the local ring of Z at the generic point of W). We then define

$$\operatorname{div}(Z, f) := \sum_W \operatorname{ord}_W(Z, f) \cdot [W],$$

where W ranges over all irreducible closed subsets of Z.

Let $\sim_{\mathrm{rat}}$ denote the subgroup of $\mathcal{Z}(X)$ generated by all cycles of the form $\operatorname{div}(Z, f)$. We finally define the *Chow group* of X to be $\mathrm{CH}(X) := \mathcal{Z}(X)/\sim_{\mathrm{rat}}$. It is clear that this quotient respects the two gradings: $\mathrm{CH}(X) = \bigoplus_{i=1}^{\dim(X)} \mathrm{CH}^i(X) = \bigoplus_{i=1}^{\dim(X)} \mathrm{CH}_i(X)$ with obvious notation.

1.2.2. Exercise. Complete the following to check that the definition makes sense.

(a) Let R be a domain, and let $g, h \in R$. Show that $\operatorname{len}_R(R/g) - \operatorname{len}_R(R/h)$ depends only on the element g/h of the fraction field of R.
(b) Show that $\operatorname{ord}_W(Z, f) = 0$ for all but finitely many W.

1.2.3. Remark. If $\mathcal{O}_{X,W}$ is regular (e.g. if X is smooth), then it is a DVR, and $\operatorname{ord}_W(Z, f)$ is just the valuation of f.

Having defined Chow groups, we briefly study two extreme cases: CH^0 and CH_0.

1.2.4. Example. It is immediate from the definition that $\mathrm{CH}^0(X)$ is still the free Abelian group on the set of connected components of X.

1.2.5. Proposition. *Assume X is proper and k is algebraically closed. Then the homomorphism* $\deg\colon \mathcal{Z}_0(X) \to \mathbb{Z}$ *which sums the coefficients of a 0-cycle is well defined on* $\mathrm{CH}_0(X)$. *In particular, we get a (non-canonical) decomposition* $\mathrm{CH}_0(X) = \mathrm{CH}_0(X)^{\deg=0} \oplus \mathbb{Z}$.

When k is not algebraically closed, the second sentence remains true after modifying the definition of deg to take into account the residue degree of a closed point; the last sentence remains true under the assumption that $X(k) \neq \varnothing$.

Proof sketch. (fill in) □

1.2.6. Example. It is easy to see that $\mathrm{CH}_0(\mathbb{P}^1)^{\deg=0} = 0$.

As the following example shows, $\mathrm{CH}_0(X)^{\deg=0}$ can have an extremely rich structure. In some sense, CH_0 is the most complicated piece of the Chow group.

1.2.7. Example. Let E be an elliptic curve over an algebraically closed field k, with point at infinity $O \in E(k)$. The theory of elliptic curves (Silverman, Chapter III, Proposition 3.4) shows that the map

$$\begin{aligned} E(k) &\longrightarrow \mathrm{CH}_0(E)^{\deg=0} \\ P &\longmapsto [\{P\}] - [\{O\}] \end{aligned}$$

is an isomorphism, where $E(k)$ is given the "chord and tangent" group law (with identity element O). A similar statement is true for a higher-genus smooth proper curve C: We have $\mathrm{CH}_0(C)^{\deg=0} \cong \mathrm{Jac}_C(k)$, where Jac_C is the Jacobian variety of C. These statements also remain true when k is not assumed to be algebraically closed (see an exercise for the case of elliptic curves). In particular, $\mathrm{CH}_0(C)$ is usually not finitely generated for a curve C of genus ≥ 1.

1.2.8. Example. Assume k is algebraically closed, and fix $x \in k$. Then $\mathrm{div}(\mathbb{A}^1, f) = [\{x\}]$, where f is the polynomial $t - x \in k[t]$. Thus $[\{x\}] = 0$ in $\mathrm{CH}_0(\mathbb{A}^1)$, and so $\mathrm{CH}_0(\mathbb{A}^1) = 0$. The reader should have no trouble showing that, more generally, $\mathrm{CH}_0(\mathbb{A}^n) = 0$ for all $n \geq 1$ and any field k.

On the other hand, similar phenomena does not hold for general affine schemes. For example, one typically has $\mathrm{CH}_0(E \setminus \{O\}) \neq 0$ for an elliptic curve E by Example 1.2.7 and Exercise 1.4.1.

The examples above demonstrate major differences between rational equivalence and homotopy (in the topological sense): We "homotoped" a point on $\mathbb{A}^1$ out of existence by moving it to infinity, yet on an elliptic curve, no point cannot be "homotoped" to any other point!

1.3. Functorialities

In this section, we explain the basic ways to move cycles along a morphism. We first give the definitions, then state the basic properties, and finally give some explanation and examples. First is "proper pushforward":

1.3.1. Definition. Let $f\colon X \to Y$ be a proper morphism. We are going to define a homomorphism $f_*\colon \mathcal{Z}_*(X) \to \mathcal{Z}_*(Y)$, where we write $\mathcal{Z}_*$ to mean that it preserves the grading by dimension. Let Z be an irreducible closed subset of X. Since f is proper, $f(Z)$ is again a closed subset of Y (and it is automatically irreducible). Viewing Z and $f(Z)$ as integral schemes, we get an extension $k(Z) \mid k(f(Z))$ of function fields of degree $d \coloneqq [k(Z) : k(f(Z))]$. We define $f_*[Z] \coloneqq d \cdot [f(Z)]$ if d is finite, and otherwise we set $f_*[Z] \coloneqq 0$.

Next is "flat pullback":

1.3.2. Definition. Let $f\colon X \to Y$ be a flat morphism. We are going to define a homomorphism $f^*\colon \mathcal{Z}^*(Y) \to \mathcal{Z}^*(X)$, where we write $\mathcal{Z}^*$ to mean that it preserves the grading by codimension. This is straightforward: Given an integral closed subscheme W of Y, let $f^{-1}(W) \coloneqq W \times_Y X$ be its scheme-theoretic preimage (whose underlying set is $f^{-1}(|W|)$), and set $f^*[W] \coloneqq [f^{-1}(W)]$ (cycle defined by a subscheme, Definition 1.1.5).

Comment on why it preserves grading by dimension!

1.3.3. Fact.

(a) *Proper pushforward and flat pullback are functorial in the sense that* $(g \circ f)_* = g_* \circ f_*$ *and* $(g \circ f)^* = f^* \circ g^*$ *whenever these identities make sense.*

(b) *Proper pushforward and flat pullback respect rational equivalence, i.e. induce homomorphisms on* CH_* *and* CH^**, respectively.*

Proof sketch. Functoriality of proper pushforward is clear. Functoriality of flat pullback follows immediately upon showing that $f^*[W] = [f^{-1}(W)]$ for an arbitrary subscheme W of Y.

□

1.3.4. Example. Assume X is proper, and let $f\colon X \to \mathrm{Spec}(k)$ be its structure morphism. Then $f_*\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(\mathrm{Spec}(k)) \xrightarrow{\sim} \mathbb{Z}$ is just the map deg defined in Proposition 1.2.5 (and the following paragraph). Note that, if k is not algebraically closed, the formula "$f_*[Z] = [f(Z)]$" does not respect rational equivalence.

1.3.5. Example. Consider the map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ given on $\overline{k}$-points by $f(x) = x^2$. Then $f^*[\{0\}] = 2 \cdot [\{0\}]$ (check this using the definition!). Thus Definition 1.1.5 is indeed playing an important role.

The above definitions can be naïvely generalized to get rid of the "proper"/"flat" hypotheses, but the resulting map on cycles need not respect rational equivalence or preserve the desired grading; we leave the reader to come up with examples. Correctly defining pullbacks in the case of a closed embedding (the typical example of a non-flat map) turns out to be quite difficult, and is essentially equivalent to what we will discuss in §1.5.

1.4. Two useful properties

In this section we describe two useful properties of Chow groups which roughly mimic properties of singular homology. First is an "excision" (or "localization") exact sequence:

1.4.1. Exercise. Let $\iota\colon Z \hookrightarrow X$ is the inclusion of a closed subscheme.

(a) Show that the sequence

$$\mathrm{CH}_i(Z) \xrightarrow{\iota_*} \mathrm{CH}_i(X) \xrightarrow{j^*} \mathrm{CH}_i(X \setminus Z) \to 0,$$

is exact for each i, where j is the inclusion $X \setminus Z \hookrightarrow X$.

(b) Show that the exact sequence of (a) is functorial in the following sense: Given a proper/flat morphism $Y \to X$, proper pushforward/flat pullback induces a morphism of exact sequences between the excision exact sequence of $Z \hookrightarrow X$ and that of $f^{-1}(Z) \hookrightarrow Y$.[8]

Next is a "homotopy invariance" property for Chow groups.

1.4.2. Proposition. *Let* $\pi\colon E \to X$ *be an* $\mathbb{A}^n$-bundle*. That is,* X *is covered by open sets* U *such that the map* $\pi|_{\pi^{-1}(U)}$ *is isomorphic to* $\mathrm{pr}_1\colon U \times \mathbb{A}^n \to U$*. Then* $\pi^*\colon \mathrm{CH}^i(X) \to \mathrm{CH}^i(E)$ *is an isomorphism.*

[8] This is a consequence of the more general compatibility between proper pushforward and flat pullback in pullback squares. [cite]

Proof sketch. Injectivity requires tools which have not been introduced, so we only explain surjectivity; details are left as an exercise. Using the excision exact sequence, induction, and the definition of rational equivalence, we reduce to showing the surjectivity of $\mathrm{pr}_1^*\colon \mathrm{CH}^1(X) \to \mathrm{CH}^1(X \times \mathbb{A}^1)$.[9] Let R be the coordinate ring of X, and consider a codimension-1 closed subset Z of $X \times \mathbb{A}^1$, given as the vanishing locus of some height-1 prime ideal $\mathfrak{p} \subseteq R[t]$. If $\mathfrak{p} \cap R \neq 0$, then Z is pulled back from X, so we assume $\mathfrak{p} \cap R = 0$. Let K be the field of fractions of R. Then $\mathfrak{p}K[t] = f(t)K[t]$ for some $f(t) = g(t)/a$, where $g(t) \in A[t] \setminus A$ and $a \in A$. One checks that $\operatorname{div}(X \times \mathbb{A}^1, f(t)) = \pi^*\alpha + [Z]$ for some $\alpha \in \mathcal{Z}^1(X)$, which implies surjectivity. $\square$

1.5. Intersection product

There is an extra structure on the homology/cohomology of a topological space coming from cap/cup product. These operations give rise to Poincaré duality. Classically, Poincaré interpreted this duality as coming from an intersection product on simplices. We are going to do something similar: endow $\mathrm{CH}(X)$ with a ring structure defined by intersecting subvarieties, at least for sufficiently nice X. We state this as follows.

1.5.1. Fact. *There exists a ring structure on* $\mathrm{CH}(X)$ *for smooth projective* X*, the* intersection product*, with the following properties:*

(a) $\mathrm{CH}(X)$ *is a graded ring with respect to the grading by codimension, i.e.*
$$\mathrm{CH}^i(X) \cdot \mathrm{CH}^j(X) \subseteq \mathrm{CH}^{i+j}(X).$$
(b) *If* $f\colon X \to Y$ *is a flat morphism of smooth projective varieties, the pullback map* $f^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ *is a ring homomorphism.*
(c) *If* f *is as in* (b)*, the* projection formula (*or* adjunction formula) *holds:*
$$f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta.$$

Correctly defining the intersection product turns out to present major complications.

We have to define the intersection product $[Z] \cdot [Z']$ for any two integral closed subschemes Z and Z' of X. We might hope that $[Z] \cdot [Z'] \coloneqq [Z \cap Z']$ is a good definition, where $Z \cap Z'$ denotes the scheme-theoretic intersection $Z \times_X Z'$ (whose underlying set is $|Z| \cap |Z'|$). Still, this cannot work in general, because Z and Z' might not be in "general position":

1.5.2. Example. Consider two lines Z and Z' in $\mathbb{P}^2$. Typically, $Z \cap Z'$ is a single point, but in the degenerate case $Z = Z'$, the intersection is an entire line. But we know by Proposition 1.2.5 that, on $\mathbb{P}^2$, the class of a point can never equal the class of a line.

We see from this example that if Z and Z' are not in "general position", they might have an intersection of larger dimension than expected. Recalling that codimensions add when intersecting two generic linear subspaces of a finite-dimensional vector space, we formulate "general position" as follows.

[9] The injectivity of pr_1^* in this case is also not difficult to prove directly, but in trying to prove injectivity in the general case, we are thwarted by the failure of the excision exact sequence to be left-exact.

1.5.3. Definition. We say that integral closed subschemes Z and Z' of X *intersect properly* if every irreducible component W of $Z \cap Z'$ satisfies $\operatorname{codim}_X(W) = \operatorname{codim}_X(Z) + \operatorname{codim}_X(Z')$.[10]

We might hope that if Z and Z' intersect properly, then $[Z] \cdot [Z'] := [Z \cap Z']$. This turns out not to be the case; however, the following facts do hold:

1.5.4. Fact. *If Z and Z' are integral closed subschemes of X which intersect properly, then*

$$[Z] \cdot [Z'] = \sum_W n_W(Z, Z') \cdot [W]$$

for certain integers $n_W(Z, Z') \geq 1$, the intersection multiplicities*, where W runs over all irreducible components of $Z \cap Z'$.*

1.5.5. Fact. *Let Z and Z' be integral closed subschemes of X which intersect properly. Let W be an irreducible component of $Z \cap Z'$, and assume that Z and Z' are "smooth along W" in the sense that the local rings $\mathcal{O}_{Z,W}$ and $\mathcal{O}_{Z',W}$ are regular. Then $n_W(Z, Z') = \operatorname{len}_{\mathcal{O}_{Z \cap Z', W}}(\mathcal{O}_{Z \cap Z', W})$.*

See Exercise 1.7.2 for an example where one cannot use this formula for the intersection multiplicity.

Anyway, a major piece of the proof of Fact 1.5.1 is the following "moving lemma". It is the reason why, in Fact 1.5.1, we only state the existence of the ring structure on $\mathrm{CH}(X)$ for smooth projective X.

1.5.6. Fact. *Let Z and Z' be any two integral closed subschemes of X. Then there exists $\alpha \in \mathcal{Z}(X)$ such that $[Z'] \sim_{\mathrm{rat}} \alpha$ and Z and α intersect properly.*

By "intersect properly" in this more general setting, we mean that if we write $\alpha = \sum_i n_i \cdot [W_i]$ for nonzero integers n_i and pairwise distinct W_i, then each W_i intersects Z properly.

Say something about Serre's formula

Note that we do not claim that α may be taken to be of the form $[Z'']$; indeed, this need not be possible, as $[Z] \cdot [Z']$ might "only be expressible using negative coefficients". (Example of Euler characteristic?)

1.6. Correspondences

In this section, all varieties will be smooth and projective. To avoid unenlightening complications, we will also assume they are connected (all definitions and results generalize in an obvious way to disjoint unions of smooth projective varieties). For proofs of the facts in this section, see [sta25, Sections `0B0H` and `0FFZ`].

The intersection product allows us to generalize the pullback and pushforward operations to all morphisms between such varieties. The key tool is the graph of a morphism:

[10] In fact, one always has $\operatorname{codim}_X(W) \leq \operatorname{codim}_X(Z) + \operatorname{codim}_X(Z')$, as predicted by linear algebra [sta25, Lemma `0AZP`].

1.6.1. Definition. Let $f\colon X \to Y$ be a morphism of connected smooth projective varieties over k. Since f is proper, the morphism $(\mathrm{id}, f)\colon X \to X \times Y$ is a closed embedding. We define the *graph* of f to be the set $\Gamma_f \coloneqq \mathrm{Img}(\mathrm{id}, f)$. We get a cycle $[\Gamma_f] \in \mathcal{Z}_{\dim(X)}(X \times Y) = \mathcal{Z}^{\dim(Y)}(X \times Y)$.

1.6.2. Definition. Let $f\colon X \to Y$ be a morphism of connected smooth projective varieties. Let pr_1 and pr_2 denote the projection maps away from $X \times Y$. We define

(a) $f_*\colon \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y)$ by $f_*(\alpha) \coloneqq \mathrm{pr}_{2*}([\Gamma_f] \cdot \mathrm{pr}_1^*(\alpha))$.
(b) $f^*\colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ by $f^*(\beta) \coloneqq \mathrm{pr}_{1*}([\Gamma_f] \cdot \mathrm{pr}_2^*(\beta))$.

Here, we write CH_* or CH^* to mean that a map respects the grading by dimension or codimension, respectively.

What's going on here? Check for yourself that if $f\colon X \to Y$ is a map of sets, and $A \subseteq X$, then $f(A) = \mathrm{pr}_2(\Gamma_f \cap \mathrm{pr}_1^{-1}(A))$. Likewise, if $B \subseteq Y$, then $f^{-1}(B) = \mathrm{pr}_1(\Gamma_f \cap \mathrm{pr}_2^{-1}(B))$. So, the formulas of Definition 1.6.2 are very natural. In fact:

1.6.3. Fact. *Let $f\colon X \to Y$ be a morphism of connected smooth projective varieties, and let f_* and f^* be as in Definition 1.6.2.*

(a) *f_* agrees with the proper pushforward. If f is flat, then f^* agrees with the flat pullback.*
(b) *f^* is a ring homomorphism and satisfies $(g \circ f)^* = f^* \circ g^*$ whenever this makes sense.*
(c) *The projection/adjunction formula holds: $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$.*

Note also that the morphism f is not playing a very significant role in the formulas of Definition 1.6.2; they only depend on the cycle $[\Gamma_f]$. This prompts the following definitions, which are a key intermediary in the construction of categories of motives.

1.6.4. Definition. Let X and Y be connected smooth projective varieties. A *correspondence* from X to Y is an element of $\mathrm{Corr}(X, Y) \coloneqq \mathrm{CH}(X \times Y)$. Given $\gamma \in \mathrm{Corr}(X, Y)$, we define

(a) $\gamma_*\colon \mathrm{CH}(X) \to \mathrm{CH}(Y)$ by $\gamma_*(\alpha) \coloneqq \mathrm{pr}_{2*}(\gamma \cdot \mathrm{pr}_1^*(\alpha))$.
(b) $\gamma^*\colon \mathrm{CH}(Y) \to \mathrm{CH}(X)$ by $\gamma^*(\beta) \coloneqq \mathrm{pr}_{1*}(\gamma \cdot \mathrm{pr}_2^*(\beta))$.

Correspondences are to be thought of as "generalized morphisms" (generalizing the case $\gamma = [\Gamma_f]$), and as such they admit an operation of *composition*:

$$\begin{aligned} \mathrm{Corr}(Y \times Z) \times \mathrm{Corr}(X \times Y) &\longrightarrow \mathrm{Corr}(X \times Z) \\ (\delta, \gamma) &\longmapsto \delta \circ \gamma \coloneqq \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*(\gamma) \cdot \mathrm{pr}_{23}^*(\delta)), \end{aligned}$$

where the pr's are the projections away from $X \times Y \times Z$.

Again, the reader should check that this formula gives the correct thing when dealing with maps of sets.

1.6.5. Fact. *Composition of correspondences is associative. The class of the diagonal $[\Delta_X] \in \mathrm{CH}(X \times X)$ is the unit element for the operation of composition.*

1.6.6. Definition. We now define $\mathbf{Corr}_k$, the *category of (smooth projective) correspondences* over k: Its objects are smooth projective varieties over k, and $\mathrm{Hom}_{\mathbf{Corr}_k}(X, Y) \coloneqq \mathrm{Corr}(X, Y)$, with composition as above.

1.6.7. Fact. *There exist two functors*

$$\mathbf{SmProj}_k \to \mathbf{Corr}_k, \qquad \mathbf{SmProj}_k^{\mathrm{op}} \to \mathbf{Corr}_k$$

defined as follows: on objects, both are the identity; the first takes a morphism $f\colon X \to Y$ to $[\Gamma_f] \in \mathrm{Corr}(X, Y)$, and the second takes f to $[\Gamma_f^{\mathrm{t}}] \in \mathrm{Corr}(Y, X)$, where $\Gamma_f^{\mathrm{t}} \subseteq Y \times X$ is the subset obtained from Γ_f by transporting it along the isomorphism $Y \times X \cong X \times Y$ (the "transpose" of Γ_f).

1.7. Additional exercises

1.7.1. In this exercise, we will study $\mathrm{CH}^*(\mathbb{P}^n)$.

(a) Let $Z \subseteq \mathbb{P}^n$ be a closed subscheme. Show that $[Z] \neq 0$ in $\mathrm{CH}^*(\mathbb{P}^n)$. (*Hint:* Reduce to the case when Z is a point, using that intersection multiplicities are always positive.)

(b) Show that

$$\mathrm{CH}^*(\mathbb{P}^n) = \mathbb{Z}[h]/h^{n+1}$$

as a ring, where h is the class of a hyperplane. Moreover, show that any codimension-i linear subspace of $\mathbb{P}^n$ has class h^i. (*Hint:* We know $\mathrm{CH}^*(\mathbb{A}^n)$ by "homotopy invariance", Proposition 1.4.2.[11])

1.7.2. Consider $Z \coloneqq V(x_1, x_2) \cup V(x_3, x_4) \subseteq \mathbb{P}^4$. Show that $[Z] \cdot h^2$ is not equal to $[Z \cap V(x_1 - x_3, x_2 - x_4)]$, despite the fact that the latter has the appropriate dimension.

1.7.3. Say that X has the *Chow–Künneth generation property* (CKgP) if for all Y, the Künneth map

$$\mathrm{CH}(X) \otimes \mathrm{CH}(Y) \to \mathrm{CH}(X \times Y)$$

is an isomorphism. Note that $\mathbb{A}^n$ has the CKgP by "homotopy invariance".

(a) Show that $\mathbb{P}^n$ has the CKgP.

(b) Let E be a genus-1 curve. Show that the image of $\mathrm{CH}(E) \otimes \mathrm{CH}(E) \to \mathrm{CH}(E \times E)$ does not contain the class of the diagonal, hence E does not have the CKgP. (*Hint:* Use the adjunction formula.)

1.7.4. Let X be equidimensional and of positive dimension. Show that given $\alpha, \beta \in \mathcal{Z}_0(X)$, there exists $\alpha' \in \mathcal{Z}_0(X)$ such that $\alpha' \sim_{\mathrm{rat}} \alpha$ and $|\alpha'| \cap |\beta| = \varnothing$.

[11]In fact, the computation of $\mathrm{CH}^*(\mathbb{A}^n)$ only requires the "surjectivity" part of Proposition 1.4.2.

2. Sites

Jake, April 22. Notes by Hunter Handley.

Some motivation for the etale topology: Weil studied ζ-functions of smooth projective varieties X over $\mathbb{F}_q$, which naively is a generating function for $\#X(\mathbb{F}_{q^n})$ for varying $n \in \mathbb{Z}_{\geq 1}$. He conjectured that this is always a rational function satisfying some functional equation, and further there is an analog of the Riemann Hypothesis about where the zeroes of this rational function should be in $\mathbb{C}$. Weil knew that these conjectures would follow in general if we have a "sufficiently nice" cohomology theory for these types of varieties. For example, the functional equation should correspond to Poincare duality and rationality corresponding to the Lefschetz fixed point theorem. This project was carried out by Grothendieck and his minions.

Given a "sufficiently nice" topological space X, we have $H^i_{\text{sing}}(X, \mathbb{Z})$, which can be computed via sheaf cohomology of $\underline{\mathbb{Z}}_X$. The main issue for us is that the Zariski topology, which is the natural one to use for a scheme X, is far too coarse (consider that it is almost the cofinite topology in the case of a curve). In fact, one would see that $H^i_{\text{Zar}}(X, \underline{\mathbb{Z}}_X) = 0, \forall i \geq 0$. The key insight of Grothendieck was that sheaves (and hence sheaf cohomology) really do not depend much on the topological space at all, just the category of open subsets of X. Thence instead of adding new open sets, one can replace the category of open subsets of X to simply be a richer category. Changing this to be the category of all X-schemes is an option but in fact too broad: the "correct" thing to use is the category of etale X-schemes.

2.0.1. Definition. Let $\mathbf{C}$ be a category (with fiber products) and $\mathrm{Cov}(\mathbf{C})$ be a set (technically class) whose elements have the following form: $U = \{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathbf{C})$ is a set of morphisms in $\mathbf{C}$ with the same target $X \in \mathbf{C}$. We say $\mathrm{Cov}(\mathbf{C})$ is a Grothendieck topology, or that $(\mathbf{C}, \mathrm{Cov}(\mathbf{C}))$ is a site, if the following hold:

(Identity) If $f : X' \to X$ is an isomorphism, then $\{f\} \in \mathrm{Cov}(\mathbf{C})$,

(Composition) if $\{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathbf{C})$ and $\forall i, \{U_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathbf{C})$, then $\{U_{ij} \to X\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathbf{C})$,

(Restriction) if $\{f_i : U_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathbf{C})$ and we have any morphism $[\phi : Y \to X] \in \mathbf{C}$, then $\{U_i \times_X Y \to Y\}_{i \in I} \in \mathrm{Cov}(\mathbf{C})$.

The following is in super-generality, which we will not need, and will simplify:

2.0.2. Definition. Let $\mathbf{C}, \mathbf{D}$ be categories. A presheaf on $\mathbf{C}$ with values in $\mathbf{D}$ is a functor $\mathbf{C}^{op} \to \mathbf{D}$. Thence $\mathbf{PSh}(\mathbf{C}, \mathbf{D})$ is the corresponding functor category whose functors are natural transformations.

Further, if $\mathbf{C}$ is a site and $\mathbf{D}$ has products, then $\mathcal{F} \in \mathbf{PSh}(\mathbf{C}, \mathbf{D})$ is a sheaf if $\forall \{U_i \to X\}_{i \in I} \in \mathrm{Cov}(\mathbf{C})$,

$$\mathcal{F}(X) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_X U_j)$$

is an equalizer diagram in the category $\mathbf{D}$. This gives a full category $\mathbf{Sh}(\mathbf{C}, \mathbf{D}) \subset \mathbf{PSh}(\mathbf{C}, \mathbf{D})$.

We will generally only care about the case of $\mathbf{D} = \mathbf{Set}$. In this case, this says that given sections $s_i \in \mathcal{F}(U_i), \forall i$, then $\exists s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$ iff $\forall i, j$ also $s_i|_{U_i \times_X U_j} = s_j|_{U_i \times_X U_j}$. Such a global section $s \in \mathcal{F}(X)$ is unique if it exists.

Here are some basic properties (left unproven!):

(a) Sheafification: "if $\mathrm{Cov}(\mathbf{C})$ is a set or close to it," then a sheafification exists– i.e. $\mathbf{Sh}(\mathbf{C}, \mathbf{Set}) \hookrightarrow \mathbf{PSh}(\mathbf{C}, \mathbf{Set})$ has a left adjoint. This says if we have a presheaf $\mathcal{F}$, we have a sheaf $\mathcal{F}^\dagger$ with a morphism $\mathcal{F} \to \mathcal{F}^\dagger$ such that any morphism $\mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$ factors through $\mathcal{F} \to \mathcal{F}^\dagger$. The fact this works for sets gives that this (more or less) helps for abelian groups, R-modules, etc.
   - This implies $\mathbf{Sh}(\mathbf{C}, \mathbf{D})$ with $\mathbf{D} = \mathbf{Set}, \mathbf{Ab}, \mathbf{Mod}_R, \dots$ have limits and colimits.
   - This further allows us to make sense of kernels and cokernels when we care about modules or abelian groups.

(b) $\mathbf{Sh}(\mathbf{C}, \mathbf{Mod}_R)$ is an abelian category with enough injectives
   - As you learned in middle school, this is necessary for nice cohomology theories.

(c) The functor $\Gamma(U) : \mathbf{Sh}(\mathbf{C}, \mathbf{Mod}_R) \to \mathbf{Mod}_R$ sending $\mathcal{F} \mapsto \mathcal{F}(U)$ is left exact and thence allows one to define $H^i(U, \mathcal{F}) := (R^i\Gamma(U))\mathcal{F}$.

Now that we are drowning, here are some examples:

- If $\mathbf{C}$ is the category of open subsets of a topological space X and $\mathrm{Cov}(\mathbf{C})$ is what you expect, we recover the usual notion of sheaves on X.
- If $\mathbf{Top}_X := \{$category of all topological spaces with a map to $X\}$ and $\mathrm{Cov}_{\mathrm{all}}(\mathbf{C}) := \{$all sets of morphisms with the same target$\}$, we get a site $(\mathbf{Top}_X)_{\mathrm{all}}$, but this is a little silly. Instead $\mathrm{Cov}_{\mathrm{surj}}(\mathbf{C}) := \{$all jointly surjective such sets$\}$ would give $(\mathbf{Top}_X)_{\mathrm{surj}}$. We could also do $\mathrm{Cov}_{\mathrm{et}}(\mathbf{C}) := \{$all jointly surjective local homeomorphisms$\}$, and get $(\mathbf{Top}_X)_{\mathrm{et}}$. Notice the difference between this and $\mathrm{Cov}_{\mathrm{Zar}}(\mathbf{C}) := \{$all jointly surjective open embeddings$\}$ giving $(\mathbf{Top}_X)_{\mathrm{Zar}}$
- Notice that $\mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{all}}) \subsetneq \mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{surj}}) \subsetneq \mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{et}}) \subsetneq \mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{Zar}})$.
- $\mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{all}}) \cong \{*\}$ with the identity morphism
- $\mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{surj}}) \cong \mathbf{Set}$.
- In general, $\mathbf{Sh}((\mathbf{Top}_X)_{\mathrm{all}}) \ncong \mathbf{Sh}(X)$.

Onto etale morphisms: recall that a morphism $f : X \to Y$ of schemes is smooth if

- f is locally of finite presentation,
- f is flat, and
- $\forall y \in Y, X_{\overline{\kappa(y)}}$ is regular.

This is analogous to the manifold case: a map $f : M \to N$ of manifolds is smooth if $\forall x \in M, \exists U \subset M, V \subset N$ open such that $x \in U, f(U) \subset V$, and $U \cong V \times \mathbb{R}^d$ for some d. A local homeomorphism of manifolds is this with $d = 0$, i.e. the fibers are 0-dimensional. An exercise is that a k-variety X is 0-dimensional iff $X = \bigsqcup_{i \in I} \mathrm{Spec}(k_i)$ where k_i/k are finite separable extensions (note that I can be an infinite index set).

2.0.3. Definition. $f : X \to Y$ is etale iff

- f is locally of finite presentation,
- f is flat,
- and $\forall y \in Y, X_{\kappa(y)} \cong \bigsqcup_{i \in I} \mathrm{Spec}(k_i)$ with k_i/k finite separable.

2.0.4. Definition. Given a scheme X, we define

- the big etale site as $\mathbf{Sch}_X$ with jointly surjective etale morphisms,
- the small etale site as $\acute{\mathbf{E}}\mathbf{t}_X$ with jointly surjective etale morphisms,
- the big/small Nisnevich sites by replacing "etale coverings" with $\{U_i \to X\}_{i \in I}$ etale such that $\forall x \in X, \exists i \in I, u \in U_i$ such that $u_i \mapsto x$ and $\kappa(u_i) = \kappa(x)$.

In the above, one must check that the composition and base change of etale morphisms are etale. But here are some examples:

- $\{\mathrm{Spec} k' \to \mathrm{Spec} k\}$ is an etale cover iff k'/k is a finite separable extension. Further, it is Nisnevich iff $[k' : k] = 1$.
- $\{\mathrm{Spec}\mathcal{O}_L[1/n] \to \mathrm{Spec}\mathcal{O}_K[1/n]\}$ where L/K are number fields is an etale cover iff L/K is unramified outside of N.
- $\{\mathbb{A}^1_k \setminus \{0\} \to^{z \mapsto z^n} \mathbb{A}^1_k \setminus \{0\}\}$ is an etale cover iff $\mathrm{char} k \nmid n$ and Nisnevich iff $n = 1$.
- $\{\mathbb{A}^1 \setminus \{x\} \to \mathbb{A}^1, \mathbb{A}^1 \setminus \{0\} \to^{z \mapsto z^n} \mathbb{A}^1\}$ with $\mathrm{char} k \nmid n, a \neq 0$ is always etale but Nisnevich iff $a^{1/n} \in k$.

2.0.5. Lemma. *A presheaf $\mathcal{F} \colon \acute{\mathbf{E}}\mathbf{t}_{\mathrm{Spec} k} \to \mathbf{Set}$ is a sheaf iff $\forall k''/k'$ finite Galois, $\mathcal{F}(\mathrm{Spec} k') \to \mathcal{F}(\mathrm{Spec} k'')^{\mathrm{Gal}(k''/k')}$ is an isomorphism. In particular, $\mathbf{Sh}(\acute{\mathbf{E}}\mathbf{t}_{\mathrm{Spec} k}, \mathbf{Set}) \cong$discrete Gal_k-sets, so the Etale cohomology can be computed via group cohomology.*

Proof. View $\mathcal{F}$ as a contravariant functor $\{\prod_{i \in I} k_i \mid k_i/k\} \to \mathbf{Set}$. One then reduces to $\mathcal{F}$ is a sheaf iff it satisfies the sheaf property for one-element covers. Then computing the equalizer gives the points fixed under $\mathrm{Gal}(k''/k')$, since $k'' \otimes_{k'} k'' \cong \prod_{\sigma \in \mathrm{Gal}(k''/k')} k''$. $\square$

References

[sta25] The Stacks project authors, *The Stacks project*, https://stacks.math.columbia.edu, 2025.